Universidade Federal Fluminense

Constant weighted mean curvature hypersurfaces in shrinking Ricci solitons

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Tese submetida ao Programa de Pós-Graduação em Matemática da Universidade Federal Fluminense como requisito parcial para a obtenção do grau de Doutor em Matemática.

Orientador: Prof. Detang Zhou

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Igor Sampaio e Melo de Miranda

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Diante da banca examinadora composta por:

Prof. Celso Melchiades Doria - UFSC
Prof. Detang Zhou (Orientador) - UFF
Prof. Ernani Ribeiro Júnior - UFC
Prof. Ezequiel Barbosa - UFMG
Prof. Hilário Alencar da Silva - UFAL
Profa. Xu Cheng - UFF

Ata dos trabalhos finais da Comissão Examinadora da Tese de Doutorado em Matemática apresentada por Igor Sampaio e Melo de Miranda

Aos dezenove dias do mês de fevereiro de dois mil e vinte e um, reuniram-se, por videoconferência, os membros da Comissão Examinadora constituída pelos Professores Detang Zhou (Universidade Federal Fluminense); Xu Cheng (Universidade Federal Fluminense); Celso Melchiades Doria (Universidade Federal de Santa Catarina); Ernani Ribeiro Júnior (Universidade Federal do Ceará); Ezequiel Barbosa (Universidade Federal de Minas Gerais) e Hilário Alencar da Silva (Universidade Federal de Alagoas), sob a presidência do primeiro, para a prova pública de defesa da tese intitulada **"CONSTANT WEIGHTED MEAN CURVATURE HYPERSURFACES IN SHRINKING RICCI SOLITONS"**, apresentada pelo Doutorando Igor Sampaio e Melo de Miranda. A defesa da tese atende às exigências contidas no Regulamento Específico do Curso de Doutorado em Matemática da Universidade Federal Fluminense. A tese foi elaborada sob a orientação do Professor Detang Zhou. O Doutorando Igor Sampaio e Melo de Miranda fez a exposição de seu trabalho durante 50 minutos, iniciando às 15h00 e concluindo às 15h50min. A seguir, respondeu às questões formuladas pelos integrantes da Comissão Examinadora. Terminada a arguição, realizou-se a reunião da Comissão Examinadora, que apresentou parecer no sentido da aprovação do Doutorando Igor Sampaio e Melo de Miranda, considerando-se o trabalho apresentado e a forma com que se houve na apresentação da defesa da mesma. Para constar, foi lavrada a presente ata, que vai assinada pela Secretária Administrativa da Coordenação de Pós-Graduação em Matemática, pelos membros da Banca Examinadora e pelo Doutorando.

Niterói, 19 de fevereiro de 2021.

Prof. Detang Zhou
(Orientador)

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Para os devidos fins, declaro **estar ciente** do conteúdo desta **versão corrigida** elaborada em atenção às sugestões dos membros da banca examinadora na sessão de defesa do trabalho, manifestando-me **favoravelmente** ao seu encaminhamento e publicação no **Repositório Institucional da UFF**.

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DEtangZhou

Detang Zhou

Para minha amada filha Bella

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RESUMO

Na teoria do fluxo da curvatura média, um tópico de grande interesse é o estudo de possíveis singularidades desse fluxo. Em $\mathbb{R}^{n+1}$, os modelos de singularidade deste fluxo podem ser associados a hipersuperfícies chamadas f -mínimas, isto é, hipersuperfícies com curvatura média com peso nula. Alguns exemplos de hipersuperfícies f-mínimas são os self-shrinkers, self-expanders e translating solitons, que desempenham um papel importante nesta teoria pois descrevem modelos de singularidades para o fluxo de curvatura média. Nesta tese, estudamos uma generalização das hipersuperfícies f-mínimas que são chamadas de hipersuperfícies CWMC ou λ-hipersuperfícies em shrinking Ricci solitons. Provamos alguns teoremas de rigidez buscando classificar essas hipersuperfícies no shrinking Ricci soliton Gaussiano e em cilindros shrinking Ricci solitons. No caso em que o ambiente é um cilindro shrinking Ricci soliton, também estudamos conjuntos de níveis e mostramos algumas propriedades geométricas das hipersuperfícies CWMC.

ABSTRACT

In the mean curvature flow theory, a topic of great interest is to study possible singularities of this flow. In $\mathbb{R}^{n+1}$, the singularity models for this flow can be associated with hypersurfaces called f-minimal, that is, hypersurfaces with null weighted mean curvature. Some examples of f-minimal hypersurfaces are self-shrinkers, self-expanders and translating solitons, they play an important role in this theory since they describe singularity models for the mean curvature flow. In this thesis, we study a generalization of f-minimal hypersurfaces which are called CWMC hypersurfaces or λ-hypersurfaces in shrinking Ricci solitons. We prove some rigidity theorems seeking to classify these hypersurfaces in the Gaussian shrinking Ricci soliton and in the cylinder shrinking Ricci solitons. For the case the ambient is a cylinder shrinking Ricci soliton, we also study level sets and show some geometric properties of CWMC hypersurfaces.

Contents

CHAPTER 1

Introduction

Let $(\bar{M}^{n+1}, \bar{g}, f)$ be a smooth measure metric space, where $(\bar{M}^{n+1}, \bar{g})$ is a Riemannian manifold endowed with the weighted volume element $e^{-f} dv_{\bar{g}}$ and f is a smooth function on $\bar{M}^{n+1}$. Considering a hypersurface M^n in $\bar{M}^{n+1}$, the weighted mean curvature of M^n is defined by

$$H_f = H - \langle \nabla f, N \rangle$$

where H is the mean curvature of M^n and N is the normal vector of M^n. Recently, there has been a great interest in the study of hypersurfaces with constant weighted mean curvature (CWMC hypersurfaces), see [CW18], [Gua18], [COW16]. These hypersurfaces generalize self-shrinkers, self-expanders and translating solitons, which are very important since they describe singularity models for the mean curvature flow (see for instance [CM12]). From the variational point of view, CWMC hypersurfaces are critical points of the weighted area functional with respect to weighted volume-preserving compact variations (see [MR15], [BSU20]).

In the study of Hamilton's Ricci flow, gradient Ricci solitons are very important since they correspond to self-similar solutions and can also be seen as models of singularity formation in the Ricci flow. In the study of CWMC hypersurfaces, it is natural to consider gradient Ricci solitons as the ambient spaces. For instance, self-shrinkers, self-expanders and translating solitons in $\mathbb{R}^{n+1}$ can be seen as f-minimal hypersurfaces ($H_f = 0$) in shrinking, expanding and steady Ricci solitons, respectively. In this thesis, we will always consider a shrinking Ricci soliton as the ambient space.

In the first part of this thesis, we will show some results in the case the ambient space is the Gaussian shrinking Ricci soliton $(\mathbb{R}^{n+1}, \bar{g}, |x|^2/4)$. In the second part, we consider shrinking Ricci solitons in general and obtain some new geometric results for CWMC hypersurfaces in the cylinder shrinking Ricci soliton.

CWMC hypersurfaces in the Gaussian Ricci soliton

In the Gaussian shrinking Ricci soliton $(\mathbb{R}^{n+1}, \bar{g}, |x|^2/4)$, a self-shrinker M^n is a hypersurface which its mean curvature satisfies

$$H = \frac{\langle x, N \rangle}{2}.$$

In the study of the mean curvature flow theory, self-shrinkers play an important role since they describe singularity models for the mean curvature flow. Also, self-shrinkers can be characterized as critical points of the weighted area functional (see for instance [CM12])

$$F(M) = \int_M e^{-\frac{|x|^2}{4}} dv. \tag{1.1}$$

In the same sense, McGonagle-Ross [MR15] studied hypersurfaces M^n in $\mathbb{R}^{n+1}$ which are critical points of the functional (1.1) for variations $G : (-\varepsilon, \varepsilon) \times M^n \to \mathbb{R}^{n+1}$ that preserve enclosed weighted volume. These variations can be represented by functions $u : M^n \to \mathbb{R}$ defined by

$$u(x) = \langle \partial_t G(0, x), N(x) \rangle$$

such that $\int_M u\, e^{-|x|^2/4} dv = 0$, where N is the normal vector of M^n. McGonagle-Ross also showed that these hypersurfaces satisfy the following condition

$$H = \frac{\langle x, N \rangle}{2} + \lambda,$$

where λ is a constant. These hypersurfaces are known as λ-hypersurfaces or CWMC hypersurfaces. It is clear that $\lambda = H_f$, where $f = |x|^2/4$. Considering $f = |x|^2/4$, we present some examples of CWMC hypersurfaces:

Example 1.1. *Self-shrinker are CWMC hypersurfaces with $H_f = 0$.*

Example 1.2. *All hyperplanes in $\mathbb{R}^{n+1}$ are CWMC hypersurfaces with $H_f = \pm\frac{d}{2}$, where d denotes the distance from the hyperplane to the origin and the sign depends on the orientation. Indeed, let M^n be a hyperplane in $\mathbb{R}^{n+1}$ and $p_0 \in M^n$ such that $d(M^n, 0) = d(p_0, 0) = d$. This implies that $\pm d = \langle p_0, N \rangle$, where N is the normal vector of M^n. Since $M^n = \{x \in \mathbb{R}^{n+1}; \langle x, N \rangle = \pm d\}$ and the mean curvature of M^n vanishes, we have*

$$H = 0 = \frac{\langle x, N \rangle}{2} \pm \frac{d}{2}.$$

Therefore, M^n is a CWMC hypersurface with $H_f = \pm d/2$, as asserted.

Example 1.3. *The spheres $S^n_r(0)$ with $r = \sqrt{H_f^2 + 2n} - H_f$ are CWMC hypersurfaces in $\mathbb{R}^{n+1}$.*

Example 1.4. *The cylinders $S^k_r(0) \times \mathbb{R}^{n-k}$ with $r = \sqrt{H_f^2 + 2k} - H_f$ are also CWMC hypersurfaces in $\mathbb{R}^{n+1}$.*

Recently, many authors have had interest in studying classification results for CWMC hypersurfaces. In the Gaussian shrinking Ricci soliton $(\mathbb{R}^{n+1}, \bar{g}, |x|^2/4)$, Huisken [Hui90] proved that the spheres $S^n_{\sqrt{2n}}(0)$ are the only compact self-shrinkers with non-negative mean curvature of dimension $n \geq 2$. Huisken [Hui] also showed that the generalized cylinders are the only complete self-shrinkers in $\mathbb{R}^{n+1}$ with non-negative mean curvature, polynomial volume growth (there exist a constant $C > 0$ and $r_0 > 0$ such that $V(B_r(0) \cap M) \leq Cr^\alpha$, for all $r \geq r_0$ and for some $\alpha > 0$) and bounded norm of the second fundamental form. Colding-Minicozzi [CM12] generalized Huisken's classification by removing the boundness condition on the second fundamental form. Later, Rimoldi [Rim14] extended Colding-Minicozzi's theorem, replacing the polynomial volume growth by an assumption on the integrability of $|A|$, where A is the second fundamental form:

Theorem 1. *[Rim14] Let M^n be a complete embedded self-shrinker in the Gaussian shrinking Ricci soliton $\mathbb{R}^{n+1}$. If $H \geq 0$ and $|A| \in L^2_f(M^n)$, then M^n is either a hyperplane or a generalized cylinder $S^k_{\sqrt{2k}}(0) \times \mathbb{R}^{n-k}$, $1 \leq k \leq n$.*

For CWMC hypersurfaces, Q.Cheng-Wei [CW18] obtained a generalization of Colding-Minicozzi's result. More precisely, they established the following result.

Theorem 2. *[CW18] Let M^n be a complete embedded CWMC hypersurface with polynomial volume growth in the Gaussian shrinking Ricci soliton $\mathbb{R}^{n+1}$. If $H - H_f \geq 0$ and $H_f \left(trA^3(H - H_f) + \frac{|A|^2}{2} \right) \leq 0$, then M^n is either a hyperplane or a generalized cylinder $S^k_r(0) \times \mathbb{R}^{n-k}$, $1 \leq k \leq n$.*

Many of the classification results for CWMC hypersurfaces assume the polynomial volume growth condition in order to use integration techniques. Recently, there has been some papers trying to avoid this assumption by using, for instance, the Omori-Yau maximum principles ([COW16], [CP15], [PR14], etc). In [AM20], we replace the polynomial volume growth assumption in Q.Cheng-Wei's theorem by an integrability condition on the norm of the second fundamental form:

Theorem 3. *Let M^n be a complete embedded CWMC hypersurface in the Gaussian shrinking Ricci soliton $\mathbb{R}^{n+1}$. Suppose that M^n satisfies the following properties:*
(i) $H - H_f \geq 0$;
(ii) $H_f \left(trA^3(H - H_f) + \frac{|A|^2}{2} \right) \leq 0$;
(iii) $|A| \in L^2_f(M^n)$
Then M^n is either a hyperplane or $S^k_r(0) \times \mathbb{R}^{n-k}$, $1 \leq k \leq n$.

Note that Theorem 3 generalizes Theorem 1 proved by Rimoldi.

Le-Sesum [LS11] also obtained classification results known as gap theorems for self-shrinkers. They showed that if M^n is a complete embedded self-shrinker with polynomial volume growth and satisfying $|A|^2 < 1/2$, then M^n is a hyperplane passing through the origin. Cao-Li [CL13] extended this result to arbitrary codimension by proving that $|A|^2 \leq 1/2$ and polynomial volume growth imply that the hypersurface is a generalized cylinder. Later, Guang [Gua18], Q.Cheng, Ogata and Wei [COW16] proved rigidity theorems for CWMC that extended Cao-Li's theorem for codimension 1. Recently, Wei-Peng proved another generalization for CWMC hypersurfaces. More specifically, they showed the following result.

Theorem 4. *[WP19] Let M^n be a complete embedded CWMC hypersurface with polynomial volume growth in the Gaussian shrinking Ricci soliton $\mathbb{R}^{n+1}$. If the norm of the second fundamental form is bounded and*

$$|A|^2 H(H - H_f) \leq \frac{H^2}{2}$$

then M^n is either a hyperplane or a generalized cylinder $S^k_r(0) \times \mathbb{R}^{n-k}$, $1 \leq k \leq n$.

Cao-Li asked in [CL13] whether it is possible to remove the polynomial volume growth condition on this kind of result. In [AM21], we extended the result proved by Wei-Peng by removing the assumption on the boundness of the second fundamental form and by assuming a mild integrability condition of the mean curvature instead of assuming polynomial volume growth.

Theorem 5. *Let M^n be a complete embedded CWMC hypersurface in the Gaussian shrinking Ricci soliton $\mathbb{R}^{n+1}$. If $H \in L^q_f(M)$ for some even number $q \geq 2$ and*

$$|A|^2 H(H - H_f) \leq \frac{H^2}{2}$$

then M^n is either a hyperplane or a generalized cylinder $S^k_r(0) \times \mathbb{R}^{n-k}$, $1 \leq k \leq n$.

Remark. *Notice that the polynomial volume growth condition implies that $H \in L^2_f(M)$.*

Remark. *The polynomial volume growth condition appears in many of the classification theorems and it is important since there are self-shrinkers that do not satisfy this condition, for instance see [Hal12].*

As an aplication of Theorem 5, we prove in [AM21] that it is possible to replace condition (i) in Theorem 3 by $H \geq 0$, obtaining the following:

Theorem 6. *Let M^n be a complete embedded CWMC hypersurface in the Gaussian shrinking Ricci soliton $\mathbb{R}^{n+1}$. Suppose that M^n satisfies the following properties:*
(i) $H \geq 0$;
(ii) $H_f \left(trA^3(H - H_f) + \frac{|A|^2}{2} \right) \leq 0$;
(iii) $|A| \in L^2_f(M^n)$
Then M^n is either a hyperplane or $S^k_r(0) \times \mathbb{R}^{n-k}$, $1 \leq k \leq n$.

Notice that Theorem 6 not only generalizes Theorem 1 proved by Rimoldi, but it also preserves the mean convexity condition, that is $H \geq 0$, which does not happen in Theorem 3.

In [AM21], we also tried to obtain a classification by assuming the reverse inequality from Theorem 5. Assuming a similar hypothesis we proved the following.

Theorem 7. *Let M^n be a complete embedded CWMC hypersurface with polynomial volume growth in the Gaussian shrinking Ricci soliton $\mathbb{R}^{n+1}$. If*

$$|A|^2(H - H_f) \geq \frac{H}{2}$$

then M^n is either a hyperplane or a generalized cylinder $S^k_r(0) \times \mathbb{R}^{n-k}$, $1 \leq k \leq n$.

The result above is new even for self-shrinkers.

Guang [Gua18] also obtained a extension of Cao-Li's result for CWMC hypersurfaces by assuming certain boundness condition on the norm of the second fundamental form and polynomial volume growth. We replace the assumption of polynomial volume growth in Guang's result by assuming certain integrability of the second fundamental form.

Theorem 8. *Let M^n be a complete embedded CWMC hypersurface in the Gaussian shrinking Ricci soliton $\mathbb{R}^{n+1}$. If $|A| \in L^q_f(M)$ for some $q \geq 2$ and*

$$|A| \leq \frac{\sqrt{H_f^2 + 2} - |H_f|}{2},$$

then either M^n is a hyperplane or it is a generalized cylinder $S^k_r(0) \times \mathbb{R}^{n-k}$, where $1 \leq k \leq n$.

We remark that for $M^n = S^n_r(0)$, $n \geq 2$ the assumption on the upper bound is not satisfied unless $H_f = 0$.

CWMC hypersurfaces in cylinder shrinking Ricci solitons

A triple $(\bar{M}^{n+1}, \bar{g}, f)$ is said to be a gradient Ricci soliton when the following holds

$$\bar{R}ic + \bar{\nabla}\bar{\nabla} f = \lambda \bar{g}$$

for some constant λ. More specifically it is called shrinking, steady and expanding when $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$, respectively. Many authors have had interest studying CWMC hypersurfaces in different ambient spaces. For instance, X.Cheng-Mejia-Zhou obtained some results for f-minimal hypersurfaces in a cylinder shrinking Ricci soliton in [CMZ15], they classified compact f-minimal hypersurfaces with a certain condition on the norm of the second fundamental form. López [Lóp18] obtained results for CWMC in a steady Ricci solitons. Barbosa-Santana-Upadhyay [BSU20] proved some results for CWMC hypersurfaces in a shrinking Ricci soliton. They also extended the result proved by McGonagle-Ross, showing that in the case the ambient is a smooth measure metric space, CWMC hypersurfaces are critical points of the weighted area functional with respect to compact weighted volume-preserving variations. We study geometric properties and classification results for CWMC hypersurfaces in shrinking Ricci solitons.

In Theorem 3 in [VZ18], Vieira-Zhou proved that if M^n is a complete f-minimal hypersurface properly immersed in the cylinder shrinking Ricci soliton then: (a) M^n cannot lie inside the closed product $\bar{B}^{n+1-k}_{\sqrt{2(n-k)}}(0) \times S^k_{\sqrt{2(k-1)}}(0)$, unless $M^n = S^{n-k}_{\sqrt{2(n-k)}}(0) \times S^k_{\sqrt{2(k-1)}}(0)$ (see also the earlier work of Cavalcante-Espinar [CE16] for CWMC hypersurfaces in the Gaussian shrinking Ricci soliton $\mathbb{R}^{n+1}$); (b) M^n cannot lie outside the product $B^{n+1-k}_{\sqrt{2(n+1-k)}}(0) \times S^k_{\sqrt{2(k-1)}}(0)$. We generalize this result for an arbitrary shrinking Ricci soliton ambient space and as consequence obtain a new result for the cylinder shrinking Ricci soliton ambient space. In order to study this result we first study level sets of the potential function f in shrinking Ricci solitons and then prove the following result:

Theorem 9. *Let M^n be a complete CWMC hypersurface properly immersed in the cylinder shrinking Ricci soliton $\mathbb{R}^{n+1-k} \times S^k_{\sqrt{2(k-1)}}(0)$, where $n \geq 3$ and $2 \leq k \leq n-1$.*

(a) If M^n lies inside the closed product $\bar{B}^{n+1-k}_r(0) \times S^k_{\sqrt{2(k-1)}}(0)$ with $r = -|H_f| + \sqrt{H_f^2 + 2(n-k)}$, then $H_f \geq 0$ and $M^n = S^{n-k}_r(0) \times S^k_{\sqrt{2(k-1)}}(0)$.

(b) M^n cannot lie outside the product $B^{n+1-k}_r(0) \times S^k_{\sqrt{2(k-1)}}(0)$ with $r = |H_f| + \sqrt{H_f^2 + 2(n+1-k)}$.

One of the applications of the result above is that it is possible to obtain a classification result assuming certain boundness condition on the curvature. This condition is important in order to assure that the hypersurface lies inside some specific region. In Theorem 3.5 in [Gua18], Guang proved that if M^n is a compact CWMC hypersurface in the Gaussian shrinking Ricci soliton $\mathbb{R}^{n+1}$ satisfying $H_f \geq 0$ and

$$|A|^2 \leq \frac{1}{2} + \frac{H_f\left(H_f + \sqrt{H_f^2 + 2n}\right)}{2n},$$

then $M^n = S^n_r(0)$, where $r = -H_f + \sqrt{H_f^2 + 2n}$. We generalize this result by assuming an upper bound on the mean curvature (this assumption is weaker since $H^2 \leq n|A|^2$) in an arbitrary shrinking

Ricci soliton ambient space. In particular, we obtain a new result for the cylinder shrinking Ricci soliton ambient space.

Theorem 10. *Let M^n be a compact CWMC hypersurface immersed in the cylinder shrinking Ricci soliton $\mathbb{R}^{n-k+1} \times S^k_{\sqrt{2(k-1)}}(0)$, where $n \geq 3$ and $2 \leq k \leq n-1$. If $H_f \geq 0$ and*

$$H \leq \frac{H_f + \sqrt{H_f^2 + 2(n-k)}}{2},$$

then $M^n = S_r^{n-k}(0) \times S^k_{\sqrt{2(k-1)}}(0)$, where $r = -H_f + \sqrt{H_f^2 + 2(n-k)}$.

Note that the assumption on the upper bound of H is sharp because equality holds for $M^n = S_r^{n-k}(0) \times S^k_{\sqrt{2(k-1)}}(0)$, where $r = -H_f + \sqrt{H_f^2 + 2(n-k)}$.

In Theorem 1.2 in [COW16], Q.Cheng-Ogata-Wei proved that if M^n is a CWMC hypersurface in the Gaussian shrinking Ricci soliton $\mathbb{R}^{n+1}$ with polynomial volume growth and satisfying

$$\left(H - \frac{H_f}{2}\right)^2 \geq \frac{H_f^2}{4} + \frac{n}{2},$$

then $M^n = S_r^n(0)$, where $r = -H_f + \sqrt{H_f^2 + 2n}$. We generalize this result to any ambient space which is a smooth measure metric spaces. In particular, we recover the result of Q.Cheng-Ogata-Wei (see Corollary 16) and we obtain a new result for the cylinder shrinking Ricci soliton ambient space.

Theorem 11. *Let M^n be a complete CWMC hypersurface properly immersed in the cylinder shrinking Ricci soliton $\mathbb{R}^{n+1-k} \times S^k_{\sqrt{2(k-1)}}(0)$, where $n \geq 3$ and $2 \leq k \leq n-1$. Then*

$$\inf H < \frac{H_f + \sqrt{H_f^2 + 2(n+1-k)}}{2}.$$

Another way to see the result above is that a complete CWMC hypersurface in the cylinder shrinking Ricci soliton cannot satisfy certain lower bound condition on the mean curvature, otherwise it would lie outside certain region which would not be possible.

This thesis is organized as follow: First we establish basic notations and definitions that are important for this work. Secondly, we present results when the ambient space is the Gaussian shrinking Ricci soliton. Then, we consider more general ambient spaces and show results when the ambient is a cylinder shrinking Ricci soliton. In the last chapter, we present further results related to these ambient spaces.

CHAPTER 2

Preliminaries

In this chapter we describe notations, conventions, definitions and examples which are important for this work.

Smooth measure metric spaces Let (M, g) be a Riemannian manifold and let f be a smooth function on M. The triple (M, g, f) is called a smooth measure metric space. The measure $e^{-f}dvol$ is called weighted volume. If u and v are functions on M^n the L^2_f inner product u and v is defined by

$$\langle u, v\rangle_{L^2_f(M)} = \int_M uve^{-f}$$

and the L^p_f norm of u is defined by

$$|u|_{L^p_f(M)} = \left(\int_M |u|^p e^{-f}\right)^{\frac{1}{p}},$$

where $p \geq 1$. The operator

$$\Delta_f = \Delta - \langle \nabla f, \nabla \cdot \rangle$$

is called weighted Laplacian. It is well known that the weighted Laplacian is a densely defined self-adjoint operator in L^2_f, that is, if u and v are smooth functions on M and v has compact support, then

$$\int_M (\Delta_f u)\, ve^{-f} = -\int_M \langle \nabla u, \nabla v\rangle\, e^{-f}.$$

The Bakry-Émery-Ricci curvature is defined by

$$Ric_f = Ric + \nabla\nabla f.$$

The triple (M, g, f) is called a gradient Ricci soliton if

$$Ric_f = \lambda g,$$

where λ is a constant. If λ is positive, zero or negative it is called shrinking, steady or expanding, respectively.

Hypersurfaces in smooth measure metric spaces. Let $\left(\bar{M}^{n+1}, \bar{g}, f\right)$ be a smooth measure metric space and let M^n be a hypersurface of $\bar{M}^{n+1}$. The second fundamental form is defined by

$$A\,(u, v) = \langle \bar{\nabla}_u v, N\rangle,$$

where u and v are vector fields on M^n, $\bar{\nabla}$ is the Riemannian connection of $\bar{M}^{n+1}$ and N is the unit normal vector. The mean curvature vector is defined by

$$\vec{H} = -HN,$$

where $H = -\mathrm{tr}_{M^n} A$. The weighted mean curvature vector is defined by

$$\vec{H}_f = \vec{H} + (\bar{\nabla} f)^{\perp}.$$

The weighted mean curvature is defined by

$$\vec{H}_f = -H_f N.$$

A hypersurface is said to have constant weighted mean curvature (CWMC) if the weighted mean curvature is constant. Note that (M^n, g, f) is also a smooth measure metric space with weighted measure $e^{-f} dvol_M$ and weighted Laplacian

$$\Delta_f = \Delta - \langle \nabla f, \nabla \cdot \rangle,$$

where $g = \bar{g}|M$, ∇ is the Riemannian connection of M^n and Δ is the Laplacian of M^n.

Example 2.1. *Let $\bar{M}^{n+1} = \mathbb{R}^{n+1}$, $\bar{g} = \bar{g}_{can}$ and $f(x) = \frac{|x|^2}{4}$. The triple $\left(\mathbb{R}^{n+1}, \bar{g}, f\right)$ is a gradient shrinking Ricci soliton with Bakry-Émery-Ricci curvature*

$$\bar{Ric}_f = \frac{1}{2}\bar{g}.$$

A hypersurface M^n of $\mathbb{R}^{n+1}$ is a CWMC hypersurface if and only if

$$H = \frac{\langle x, N \rangle}{2} + H_f.$$

Note that $H_f = 0$ if and only if M^n is a self-shrinker. The weighted volume of M^n is $e^{-\frac{|x|^2}{4}} dvol_M$ and the weighted Laplacian of M^n is given by

$$\Delta_f = \Delta - \frac{1}{2}\langle x, \nabla \cdot \rangle.$$

Note that Δ_f is the operator $\mathcal{L}$ introduced by Colding and Minicozzi [CM12]. We say that $\left(\mathbb{R}^{n+1}, \bar{g}, f\right)$ is the Gaussian shrinking Ricci soliton.

Example 2.2. *Let $\bar{M}^{n+1} = \mathbb{R}^{n+1}$, $\bar{g} = \bar{g}_{can}$ and $f(x) = -\frac{|x|^2}{4}$. The triple $\left(\mathbb{R}^{n+1}, \bar{g}, f\right)$ is a gradient expanding Ricci soliton with Bakry-Émery-Ricci curvature*

$$\bar{Ric}_f = -\frac{1}{2}\bar{g}.$$

A hypersurface M^n of $\mathbb{R}^{n+1}$ is a CWMC hypersurface if and only if

$$H = -\frac{\langle x, N \rangle}{2} + H_f.$$

Note that $H_f = 0$ if and only if M^n is a self-expander. The weighted volume of M^n is $e^{\frac{|x|^2}{4}} dvol_M$ and the weighted Laplacian of M^n is given by

$$\Delta_f = \Delta + \frac{1}{2}\langle x, \nabla \cdot \rangle.$$

Example 2.3. *Let* $\bar{M}^{n+1} = \mathbb{R}^{n+1}$, $\bar{g} = \bar{g}_{can}$ *and* $f(x) = \langle a, x\rangle$, *where* $a \in \mathbb{R}^{n+1}$. *The triple* $\left(\mathbb{R}^{n+1}, \bar{g}, f\right)$ *is a gradient steady Ricci soliton with Bakry-Émery-Ricci curvature*

$$\bar{R}ic_f = 0.$$

A hypersurface M^n *of* $\mathbb{R}^{n+1}$ *is a CWMC hypersurface if and only if*

$$H = \langle a, N\rangle + H_f.$$

Note that $H_f = 0$ *if and only if* M^n *is a translating soliton. The weighted volume of* M^n *is* $e^{-\langle a,x\rangle} dvol_M$ *and the weighted Laplacian of* M^n *is given by*

$$\Delta_f = \Delta - \langle a, \nabla\cdot\rangle .$$

.

Example 2.4. *Let* $\bar{M}^{n+1} = \mathbb{R}^{n+1-k} \times S^k_{\sqrt{2(k-1)}}$ *with product metric* $\bar{g}$ *and potential function* $f(x, y) = \frac{|x|^2}{4}$. *Here* x *is the position vector in* $\mathbb{R}^{n-k+1}$ *and* y *is the position vector in* $\mathbb{R}^{k+1}$. *The triple* $\left(\mathbb{R}^{n+1-k} \times S^k_{\sqrt{2(k-1)}}, \bar{g}, f\right)$ *is a gradient shrinking Ricci soliton with Bakry-Émery-Ricci curvature*

$$\begin{aligned}\bar{R}ic_f &= \bar{\nabla}\bar{\nabla} f + \bar{R}ic \\ &= \frac{1}{2} g_{\mathbb{R}^{n+1-k}} + \frac{1}{2} g_{S^k_{\sqrt{2(k-1)}}} \\ &= \frac{1}{2}\bar{g}.\end{aligned}$$

A hypersurface M^n *of* $\mathbb{R}^{n+1-k} \times S^k_{\sqrt{2(k-1)}}$ *is a CWMC hypersurface if and only if*

$$H = \frac{\langle x, N\rangle}{2} + H_f.$$

The weighted volume of M^n *is* $e^{-\frac{|x|^2}{4}} dvol_M$ *and the weighted Laplacian of* M^n *is given by*

$$\Delta_f = \Delta - \frac{1}{2}\langle x, \nabla\cdot\rangle .$$

We say that $\left(\mathbb{R}^{n+1-k} \times S^k_{\sqrt{2(k-1)}}, \bar{g}, f\right)$ *is the cylinder shrinking Ricci soliton.*

CHAPTER 3

Rigidity Theorems for CWMC hypersurfaces

In this chapter, we will prove Theorems 3, 5, 6, 7 and 8. We will also see some consequences and applications. Most of these results can be found in [AM20] and [AM21], joint works with Saul Ancari.

For CWMC hypersurfaces, the following equations will be needed to prove the classification theorems.

Lemma 1. *If M^n is a CWMC hypersurface in the Gaussian shrinking Ricci soliton $\mathbb{R}^{n+1}$, then*

$$\Delta_f(H-H_f)+\left(|A|^2-\frac{1}{2}\right)(H-H_f)=\frac{H_f}{2} \tag{3.1}$$

and

$$\Delta_f|A|+\left(|A|^2-\frac{1}{2}\right)|A|=\frac{|\nabla A|^2-|\nabla|A||^2}{|A|}-\frac{H_f trA^3}{|A|}. \tag{3.2}$$

Proof. Let $p\in M^n$ and consider a local orthonormal frame such that $\nabla_{e_j}e_i(p)=0$, $1\leq i,j\leq n$. Denote as $h_{ij}=A(e_i,e_j)$. Since M^n is a CWMC hypersurface

$$\begin{aligned} e_i(H) &= \frac{1}{2}\langle x,\bar{\nabla}_{e_i}N\rangle \\ &= -\frac{1}{2}\left\langle x,\sum_{k=1}^n h_{ik}e_k\right\rangle. \end{aligned}$$

Therefore, the following holds at p

$$\begin{aligned} e_je_i(H) &= -\sum_{k=1}^n h_{ikj}\frac{\langle x,e_k\rangle}{2}-\frac{h_{ij}}{2}-\sum_{k=1}^n h_{ik}\frac{\langle x,\bar{\nabla}_{e_j}e_k\rangle}{2} \\ &= -\sum_{k=1}^n h_{ijk}\frac{\langle x,e_k\rangle}{2}-\frac{h_{ij}}{2}-\sum_{k=1}^n h_{ik}h_{jk}\frac{\langle x,N\rangle}{2} \\ &= -\sum_{k=1}^n h_{ijk}\frac{\langle x,e_k\rangle}{2}-\frac{h_{ij}}{2}-(H-H_f)\sum_{k=1}^n h_{ik}h_{jk}. \end{aligned} \tag{3.3}$$

To prove (3.1), consider $i=j$ and summing in i we have

$$\begin{aligned} \Delta H &= -\sum_{i,k=1}^n h_{iik}\frac{\langle x,e_k\rangle}{2}-\frac{1}{2}\sum_{i=1}^n h_{ii}-(H-H_f)\sum_{i,k=1}^n h_{ik}^2 \\ &= \frac{\langle x,\nabla H\rangle}{2}+\frac{H}{2}-(H-H_f)|A|^2. \end{aligned}$$

Since $\Delta_f H = \Delta H - \frac{1}{2}\langle x, \nabla H\rangle$, we have

$$\begin{aligned}\Delta_f H &= \frac{H}{2} - (H - H_f)|A|^2 \\ &= \frac{H}{2} - (H - H_f)\left(|A|^2 - \frac{1}{2}\right) - \frac{1}{2}(H - H_f) \\ &= -(H - H_f)\left(|A|^2 - \frac{1}{2}\right) + \frac{H_f}{2}\end{aligned}$$

and the first equation holds.

To prove (3.2), recall that the Simon's equation (see for instance [CMZ15]) states that

$$\frac{1}{2}\Delta|A|^2 = |\nabla A|^2 - \langle A, \text{Hess } H\rangle - H\text{tr}A^3 - |A|^4.$$

From equation (3.3), we have

$$h_{ij}H_{ij} = -\sum_{k=1}^{n} h_{ij}h_{ijk}\frac{\langle x, e_k\rangle}{2} - \frac{h_{ij}^2}{2} - (H - H_f)\sum_{k=1}^{n} h_{ik}h_{jk}h_{ij}.$$

Since $\nabla|A|^2 = 2\sum_{i,j,k=1}^{n} h_{ij}h_{ijk}e_k$, summing in i and j we get

$$\sum_{i,j=1}^{n} h_{ij}H_{ij} = -\frac{\langle x, \nabla|A|\rangle}{4} - |A|^2 - (H - H_f)\text{tr}A^3.$$

Combining the equation above and the Simon's equation, we obtain

$$\begin{aligned}\frac{1}{2}\Delta_f|A|^2 &= \frac{|A|^2}{2} + (H - H_f)\text{tr}A^3 - H\text{tr}A^3 - |A|^4 + |\nabla A|^2 \\ &= \left(\frac{1}{2} - |A|^2\right)|A|^2 - H_f\text{tr}A^3 + |\nabla A|^2.\end{aligned}$$

Since $\Delta_f|A|^2 = 2|A|\Delta_f|A| + 2|\nabla|A||^2$, from the equation above we get that

$$\Delta_f|A| = -\left(|A|^2 - \frac{1}{2}\right)|A| + \frac{|\nabla A|^2 - |\nabla|A||^2}{|A|} - \frac{H_f\text{tr}A^3}{|A|}.$$

□

Using the lemma above, we prove the first main result of this thesis.

Theorem 12. *(Theorem 3) Let M^n be a complete embedded CWMC hypersurface in the Gaussian shrinking Ricci soliton $\mathbb{R}^{n+1}$. Suppose that M^n satisfies the following properties:*
(i) $H - H_f \geq 0$;
(ii) $H_f\left(trA^3(H - H_f) + \frac{|A|^2}{2}\right) \leq 0$;
(iii) $|A| \in L_f^2(M^n)$
Then M^n is either a hyperplane or $S_r^k(0) \times \mathbb{R}^{n-k}$, $1 \leq k \leq n$.

Proof For $H_f \leq 0$, from equation (3.1) we have

$$\Delta_f(H - H_f) + (|A|^2 - \frac{1}{2})(H - H_f) = \frac{H_f}{2} \leq 0. \tag{3.4}$$

Since $H-H_f\geq 0$, by the maximum principle we can conclude that either $H-H_f=0$ or $H-H_f>0$. If $H-H_f=0$, then from (3.4) we conclude that $H_f=0$, which implies that M^n is a self-shrinker. Moreover, Colding-Minicozzi proved in [CM12] that a self-shrinker such that $H=0$ has to be a hyperplane. If $H_f>0$ and $H-H_f=0$ at some point $p\in M^n$, from hypothesis (ii) we have

$$0\geq H_f\left(\mathrm{tr}A^3(H-H_f)+\frac{|A|^2}{2}\right)=\frac{H_f|A|^2}{2}$$

at $p\in M^n$. This implies that $|A|(p)=0$, but this contradicts the fact that $H(p)>0$. To conclude the proof, we only need to study the case $H-H_f>0$.

In this case, we will prove that either $|A|=0$ or $|A|=C(H-H_f)$, for $C>0$. Consider the functions $u=H-H_f$ and $v=\sqrt{|A|^2+\varepsilon}$. Computing $\Delta_f u$ and $\Delta_f v$, we get

$$\Delta_f u+\left(|A|^2-\frac{1}{2}\right)u=\frac{H_f}{2} \tag{3.5}$$

and

$$\Delta_f v+\left(|A|^2-\frac{1}{2}\right)v=\frac{|\nabla A|^2-|\nabla v|^2}{v}+\left(|A|^2-\frac{1}{2}\right)\frac{\varepsilon}{v}-\frac{H_f\mathrm{tr}A^3}{v}.$$

Since

$$\frac{|\nabla A|^2-|\nabla v|^2}{v}\geq 0,\quad \Delta_f v+\left(|A|^2-\frac{1}{2}\right)v\geq -\frac{\varepsilon}{2v}-\frac{H_f\mathrm{tr}A^3}{v}. \tag{3.6}$$

Let us consider $w=\frac{v}{u}$. Thus, from (3.5) we get

$$\begin{aligned}\Delta_f v &= w\Delta_f u+2\langle\nabla w,\nabla u\rangle+u\Delta_f w\\ &= w\left(\frac{H_f}{2}-|A|^2u+\frac{u}{2}\right)+2\langle\nabla w,\nabla u\rangle+u\Delta_f w.\end{aligned}$$

By (3.6), we have

$$u\Delta_f w\geq -\frac{1}{v}\left(\frac{\varepsilon}{2}+H_f\mathrm{tr}A^3\right)-\frac{H_f w}{2}-2\langle\nabla w,\nabla u\rangle.$$

Using hypothesis (ii), it is possible to conclude that

$$\frac{u}{v}\left(-\frac{\varepsilon}{2}-H_f\mathrm{tr}A^3\right)-\frac{vH_f}{2}\geq -\frac{\varepsilon H}{2v}. \tag{3.7}$$

From the inequality above, we obtain

$$\Delta_f w\geq -\frac{\varepsilon H}{2vu^2}-2\langle\nabla w,\nabla\log u\rangle. \tag{3.8}$$

For a function $\varphi\in C_0^\infty(M^n)$, using integration by parts and (3.8), we get

$$\begin{aligned}\int_{M^n}\varphi^2|\nabla w|^2e^{-f} &= -\int_{M^n}\varphi^2 w\Delta_f w e^{-f}-\int_{M^n}2\varphi w\langle\nabla\varphi,\nabla w\rangle e^{-f}\\ &\leq 2\int_{M^n}\varphi^2 w\langle\nabla w,\nabla\log u\rangle e^{-f}+\frac{\varepsilon}{2}\int_{M^n}\frac{\varphi^2 wH}{vu^2}e^{-f}-\int_{M^n}2\varphi w\langle\nabla\varphi,\nabla w\rangle e^{-f}\\ &= 2\int_{M^n}\langle\varphi\nabla w,\varphi w\nabla\log u-w\nabla\varphi\rangle e^{-f}+\frac{\varepsilon}{2}\int_{M^n}\frac{\varphi^2H}{u^3}e^{-f}\\ &\leq \frac{1}{2}\int_{M^n}\varphi^2|\nabla w|^2e^{-f}+2\int_{M^n}w^2|\varphi\nabla\log u-\nabla\varphi|^2e^{-f}+\frac{\varepsilon}{2}\int_{M^n}\frac{\varphi^2H}{u^3}e^{-f}.\end{aligned}$$

Therefore,

$$\int_{M^n}\varphi^2|\nabla w|^2e^{-f}\le 4\int_{M^n}w^2|\varphi\nabla\log u-\nabla\varphi|^2e^{-f}+\varepsilon\int_{M^n}\frac{\varphi^2 H}{u^3}e^{-f}.$$

Choosing $\varphi=\psi u$, $\psi\in C_0^\infty(M^n)$, we have

$$\int_{M^n}\psi^2u^2|\nabla w|^2e^{-f}\ \le\ 4\int_{M^n}v^2|\nabla\psi|^2e^{-f}+\varepsilon\int_{M^n}\psi^2e^{-f}+\varepsilon H_f\int_{M^n}\frac{\psi^2}{u}e^{-f}.$$

For $H_f\ge 0$, choosing $\varepsilon=0$ we obtain

$$\int_{M^n}\psi^2u^2|\nabla w|^2e^{-f}\ \le\ 4\int_{M^n}v^2|\nabla\psi|^2e^{-f}.$$

In the case that M^n is compact, choose $\psi=1$. Otherwise, consider a sequence $\psi_k\in C_0^\infty(M^n)$, such that $\psi_k=1$ in $B_k^M(p)$, $\psi_k=0$ in $M^n\setminus B_{2k}^M(p)$ and $|\nabla\psi_k|\le 1/k$ for every k, we have

$$\begin{aligned}\int_{M^n}\psi_k^2u^2|\nabla w|^2e^{-f}\ &\le\ 4\int_{B_{2k}^M(p)\setminus B_k^M(p)}v^2|\nabla\psi_k|^2e^{-f}\\ &\le\ \frac{4}{k^2}\int_{B_{2k}^M(p)\setminus B_k^M(p)}v^2e^{-f}\\ &=\ \frac{4}{k^2}\int_{B_{2k}^M(p)\setminus B_k^M(p)}|A|^2e^{-f}.\end{aligned}$$

By the monotone convergence theorem and hypothesis (iii), we get

$$\int_{M^n}u^2\left|\nabla\left(\frac{|A|}{H-H_f}\right)\right|^2e^{-f}=0,$$

which implies that $|A|=C(H-H_f)$, for a constant $C>0$.

For $H_f<0$, we have

$$\int_{M^n}\psi^2u^2|\nabla w|^2e^{-f}\le 4\int_{M^n}v^2|\nabla\psi|^2e^{-f}+\varepsilon\int_{M^n}\psi^2e^{-f}. \tag{3.9}$$

In the non-compact case, consider a sequence $\psi_k\in C_0^\infty(M^n)$, such that $\psi_k=1$ in $B_k^M(p)$, $\psi_k=0$ in $M^n\setminus B_{2k}^M(p)$ and $|\nabla\psi_k|\le 1/k$ for every k, hence we get

$$\begin{aligned}\int_{M^n}\psi_k^2u^2|\nabla w|^2e^{-f}\ &\le\ 4\int_{M^n}v^2|\nabla\psi_k|^2e^{-f}+\varepsilon\int_{M^n}\psi_k^2e^{-f}\\ &\le\ \frac{4}{k^2}\int_{B_{2k}^M(p)\setminus B_k^M(p)}|A|^2e^{-f}+\frac{4\varepsilon}{k^2}\int_{B_{2k}^M(p)\setminus B_k^M(p)}e^{-f}+\varepsilon\int_{B_{2k}^M(p)}e^{-f}.\end{aligned}$$

Choosing $\varepsilon=\left(k\int_{B_{2k}^M(p)}e^{-f}\right)^{-1}$, we have

$$\int_{M^n}\psi_k^2u^2|\nabla w|^2e^{-f}\le\frac{4}{k^2}\int_{B_{2k}^M(p)\setminus B_k^M(p)}|A|^2e^{-f}+\frac{4}{k^3}+\frac{1}{k}.$$

Hence, by hypothesis (iii) we obtain

$$\lim_{k\to\infty}\int_{M^n}\psi_k^2u^2|\nabla w|^2e^{-f}=0.$$

If the set

$$\mathcal{A} = \{p \in M^n; |A|(p) = 0\}$$

is not empty, consider $\mathcal{B} = M^n \setminus \mathcal{A}$. Since $\mathcal{B}$ is an open set, let $p \in \mathcal{B}$ and $B_r^M(p) \subset \mathcal{B}$. For k sufficiently large, $B_r^M(p) \subset \mathrm{supp}\psi_k$ and $\psi_k = 1$ in $B_r^M(p)$. Hence

$$\lim_{k\to\infty} \int_{B_r^M(p)} u^2 |\nabla w|^2 e^{-f} = 0.$$

By the dominated convergence theorem, we conclude that $|A|/(H - H_f)$ is constant in $B_r^M(p)$. Since p is arbitrary, it is possible to conclude that $|A|/(H - H_f)$ is constant in $\mathcal{B}$. Since $\mathcal{A} \neq \emptyset$, using a continuity argument, we conclude that $|A| = 0$. If $\mathcal{A} = \emptyset$, by the dominated convergence theorem

$$\int_{M^n} u^2 \left|\nabla \left(\frac{|A|}{H - H_f}\right)\right|^2 e^{-f} = 0,$$

which implies $|A| = C(H - H_f)$ for a constant $C > 0$.

In the case that M^n is compact, choosing $\psi = 1$ in (3.9) we have

$$\int_{M^n} u^2 |\nabla w|^2 e^{-f} \leq \varepsilon \int_{M^n} e^{-f}.$$

Therefore

$$\lim_{\varepsilon\to 0} \int_{M^n} u^2 |\nabla w|^2 e^{-f} = 0.$$

Using a similar argument as before we also conclude that $|A| = C(H - H_f)$ for a constant $C > 0$.

Hence, when $H - H_f > 0$, we conclude that either $|A| = 0$ or $|A| = C(H - H_f)$ for a constant $C > 0$. If $|A| = 0$, then M^n is a hyperplane. Otherwise, since $|A| = C(H - H_f)$, it holds

$$\begin{aligned}
\Delta_f |A| &= \frac{|A|}{H - H_f}\Delta_f(H - H_f)\\
&= \frac{|A|H_f}{2(H - H_f)} + \left(\frac{1}{2} - |A|^2\right)|A|.
\end{aligned}$$

On the other hand, notice that

$$\Delta_f |A| = \left(\frac{1}{2} - |A|^2\right)|A| + \frac{|\nabla A|^2 - |\nabla|A||^2}{|A|} - \frac{H_f \mathrm{tr} A^3}{|A|}.$$

Hence, from the equations above we have

$$\begin{aligned}
\frac{|\nabla A|^2 - |\nabla|A||^2}{|A|} &= \frac{|A|H_f}{2(H - H_f)} + \frac{H_f \mathrm{tr} A^3}{|A|}\\
&= \frac{H_f}{(H - H_f)|A|}\left(\mathrm{tr} A^3 (H - H_f) + \frac{|A|^2}{2}\right).
\end{aligned}$$

Using the hypothesis (ii) and the equality above, we conclude that

$$|\nabla|A|| = |\nabla A|. \tag{3.10}$$

Fixing $p \in M^n$ and $\{E_i\}_{1 \le i \le n}$ an orthonormal basis for T_pM^n, (3.10) implies that for each k there exists a constant C_k such that

$$h_{ijk} = C_k h_{ij}$$

for all i, j. Considering a base such that $h_{ij} = \lambda_i \delta_{ij}$, by the Codazzi equation, we have

$$h_{ijk} = 0$$

unless $i = j = k$. If $\lambda_i \neq 0$ and $i \neq j$, then

$$0 = h_{iij} = C_j \lambda_i.$$

Hence, it follows that $C_j = 0$. Thus, if the rank of the matrix (h_{ij}) is at least two at p, then $\nabla A(p) = 0$. To show that $\nabla A = 0$, let us fix $q \in M^n$ and suppose that $\lambda_1(q)$ and $\lambda_2(q)$ are the largest eigenvalues of $(h_{ij})(q)$. Define the following set

$$\Lambda = \{q \in M^n; \lambda_1(q) = \lambda_1(p), \lambda_2(q) = \lambda_2(p)\}.$$

Using the continuity of the $\lambda_i's$, it is possible to prove that the set Λ is open and closed. Since $p \in \Lambda$ and M^n is connected, $\Lambda = M^n$. Therefore, $\nabla A = 0$ everywhere on M^n. Hence, M^n is an isoparametric hypersurface and by a theorem proved by Lawson in [Law69], M^n must be $S^k_r(0) \times \mathbb{R}^{n-k}$ with $2 \le k \le n$. If the rank of the matrix (h_{ij}) is one, then

$$H^2 = |A|^2 = C^2(H - H_f)^2.$$

From this equation, H must be constant. Moreover, from

$$|\nabla A| = |\nabla |A|| = C|\nabla H| = 0,$$

we conclude that M^n is isoparametric and by Lawson's result M^n must be $S^1_r(0) \times \mathbb{R}^{n-1}$. This finishes the proof of the theorem.

□

Notice that when $M^n = S^n_r(0)$ it follows that $H_f = \frac{2n-r^2}{2r}$, $H = \frac{n}{r}$, $\text{tr}A^3 = -\frac{n}{r^3}$ and $|A|^2 = \frac{n}{r^2}$ and therefore

$$\begin{aligned} \text{tr}A^3(H - H_f) + \frac{|A|^2}{2} &= -\frac{n}{r^3}\left(\frac{n}{r} - \frac{2n-r^2}{2r}\right) + \frac{n}{2r^2} \\ &= -\frac{n}{r^3}\left(\frac{r}{2}\right) + \frac{n}{2r^2} \\ &= 0. \end{aligned}$$

Hence equality holds on the condition (ii) of Theorem 3 when M^n is a sphere. The same happens with cylinders and hyperplanes.

In the sequel we shall present the proof of Theorem 5.

Theorem 13. *(Theorem 5) Let M^n be a complete embedded CWMC hypersurface in the Gaussian shrinking Ricci soliton $\mathbb{R}^{n+1}$. If $H \in L^q_f(M)$ for some even number $q \ge 2$ and*

$$|A|^2 H(H - H_f) \le \frac{H^2}{2}$$

then M^n is either a hyperplane or a generalized cylinder $S^k_r(0) \times \mathbb{R}^{n-k}$, $1 \le k \le n$.

Proof By equation (3.1), considering q as in the hypothesis and $p = q - 1$ we have

$$\begin{aligned} H^p \Delta_f H &= \frac{H^{p+1}}{2} - |A|^2 H^p (H - H_f) \\ &= H^{p-1}\left(\frac{H^2}{2} - |A|^2 H(H - H_f)\right). \end{aligned}$$

Since q is an even number, $p - 1$ is also even. Therefore, by the hypothesis

$$H^p \Delta_f H \geq 0.$$

Let $\varphi \in C_0^\infty(M^n)$, from the fact above we have

$$\begin{aligned} p\int_{M^n} \varphi^2 H^{p-1} |\nabla H|^2 e^{-f} &= -\int_{M^n} \varphi^2 H^p \Delta_f H e^{-f} - \int_{M^n} \langle \nabla\varphi^2, H^p \nabla H\rangle e^{-f} \\ &\leq -2\int_{M^n} \langle H^{\frac{p+1}{2}} \nabla\varphi, \varphi H^{\frac{p-1}{2}} \nabla H\rangle e^{-f} \\ &\leq \int_{M^n} \left(2H^q |\nabla\varphi|^2 + \frac{1}{2}\varphi^2 H^{p-1} |\nabla H|^2\right) e^{-f}. \end{aligned}$$

Therefore,

$$\left(p - \frac{1}{2}\right)\int_{M^n} \varphi^2 H^{p-1} |\nabla H|^2 e^{-f} \leq 2\int_{M^n} H^q |\nabla\varphi|^2 e^{-f}.$$

When M^n is compact, we choose $\varphi = 1$. Otherwise, consider a sequence $\varphi_j \in C_0^\infty$ such that $\varphi_j(x) = 1$ for $x \in B_j^M(p)$, $\varphi_j(x) = 0$ for $x \in M^n \setminus B_{2j}^M(p)$ and $|\nabla\varphi_j| \leq 1/j$, by the monotone convergence theorem and the assumption that $H \in L_f^q(M^n)$, we get

$$H^{p-1} |\nabla H| = 0.$$

Consider the set $\mathcal{A} = \{x \in M^n; H(x) = 0\}$. The set $M \setminus \mathcal{A}$ is open and

$$|\nabla H| = 0,$$

which implies that H is constant in $M^n \setminus \mathcal{A}$. If $\mathcal{A} \neq \emptyset$, by using continuity we conclude that $H = 0$, and by equation (3.1) M^n is a hyperplane. Otherwise, $H \neq 0$ and it is constant. Hence, $|A|^2 = \frac{H}{2(H - H_f)}$ and $|A|$ is also constant. From equation (3.2) it follows that

$$|\nabla A|^2 = \left(|A|^2 - \frac{1}{2}\right)|A|^2 + H_f \mathrm{tr} A^3. \tag{3.11}$$

On the other hand, since the Simon's equation holds, we have

$$\frac{1}{2}\Delta |A|^2 = |\nabla A|^2 - \langle A, \text{Hess } H\rangle - H \mathrm{tr} A^3 - |A|^4,$$

it follows that

$$|\nabla A|^2 = H \mathrm{tr} A^3 + |A|^4. \tag{3.12}$$

Therefore, combining (3.11) and (3.12), we obtain

$$\mathrm{tr} A^3 = \frac{|A|^2}{2(H_f - H)}.$$

From (3.12) and the equation above, we have

$$|\nabla A|^2 = \frac{H|A|^2}{2(H_f - H)} + |A|^4$$

and since $|A|^2 = \frac{H}{2(H-H_f)}$, we conclude that

$$|\nabla A|^2 = 0.$$

By Lawson's theorem, M^n must be a generalized cylinder, which finishes the proof of the theorem.

□

Notice that when $M^n = S_r^n(0)$ it follows that $H_f = \frac{2n-r^2}{2r}$, $H = \frac{n}{r}$ and $|A|^2 = \frac{n}{r^2}$ and therefore

$$\begin{aligned} |A|^2 H(H-H_f) - \frac{H^2}{2} &= \frac{n^2}{r^3}\left(\frac{n}{r} - \frac{2n-r^2}{2r}\right) - \frac{n^2}{2r^2} \\ &= \frac{n^2}{r^3}\left(\frac{r}{2}\right) - \frac{n^2}{2r^2} \\ &= 0. \end{aligned}$$

Hence equality holds on the condition Theorem 5 when M^n is a sphere. The same happens with cylinders and hyperplanes.

An immediate consequence of Theorem 5 for self-shrinkers is the following result.

Corollary 1. *Let M^n be a complete embedded self-shrinker in the Gaussian shrinking Ricci soliton $\mathbb{R}^{n+1}$. If $H \in L_f^q(M)$ for some even number $q \geq 2$ and $|A|^2 \leq \frac{1}{2}$, then M^n is either a hyperplane passing through the origin or a generalized cylinder $S^k_{\sqrt{2k}}(0) \times \mathbb{R}^{n-k}$, $1 \leq k \leq n$.*

Another consequence of Theorem 5 is the following.

Corollary 2. *Let M^n be a complete embedded CWMC hypersurface with finite weighted volume in the Gaussian shrinking Ricci soliton $\mathbb{R}^{n+1}$. If*

$$|A|^2 H(H-H_f) \leq \frac{H^2}{2},$$

then M^n is either a hyperplane or a generalized cylinder $S_r^k(0) \times \mathbb{R}^{n-k}$, $1 \leq k \leq n$.

Proof. The compact case was proved in Theorem 5, therefore we only need to consider the non-compact case. We will see that the hypothesis implies that $\sup_{x\in M^n} H < +\infty$. In fact, if $\sup_{x\in M^n} H = +\infty$, then there exists a sequence $\{p_j\}_{j\in\mathbb{N}}$ in M^n such that $H(p_j) \to +\infty$ when $j \to +\infty$. Therefore, for j large enough $H(p_j) > 0$ and $\frac{H_f}{H(p_j)} < 1$. From the hypothesis, we have

$$|A|^2(p_j) \leq \frac{1}{2}\left(1 - \frac{H_f}{H(p_j)}\right)^{-1}.$$

Since $\left(1 - \frac{H_f}{H(p_j)}\right)^{-1} \to 1$ when $j \to +\infty$, then the right side of the inequality above is bounded which implies that $|A|^2(p_j)$ is bounded. Therefore, $H^2(p_j)$ is also bounded, but this contradicts the assumption that $H(p_j) \to +\infty$. Using a similar argument, it is possible to prove that $\inf_{x\in M^n} H > -\infty$. Hence, the mean curvature of M^n is bounded. The result then follows by applying Theorem 5. □

Recently, Q.Cheng-Wei [CW18] proved that a CWMC hypersurface with constant mean curvature must be locally isometric to a generalized cylinder. Using the same argument that was used at the end of the proof of Theorem 5, it is possible to conclude the following corollary.

Corollary 3. *If M^n is a complete embedded CWMC hypersurface with constant mean curvature in the Gaussian shrinking Ricci soliton $\mathbb{R}^{n+1}$, then M^n is either a hyperplane or a generalized cylinder $S^k_r(0) \times \mathbb{R}^{n-k}$, $1 \leq k \leq n$.*

Proof. If H is constant, from equation (3.2) we have

$$\frac{H^2}{2} = |A|^2 H(H - H_f).$$

The rest of the proof follows as in the proof of Theorem 5. □

Remark. *Using the argument from the proof of Wei-Peng's theorem [WP19], one can also obtain the same result.*

Notice that in the proof of Theorem 5, one can obtain a condition which implies that M^n must be a hyperplane:

Corollary 4. *Let M^n be a complete embedded CWMC hypersurface in the Gaussian shrinking Ricci soliton $\mathbb{R}^{n+1}$. If $H \in L^q_f(M^n)$ for some even number $q \geq 2$ and*

$$0 \leq H \leq H_f,$$

then M^n is a hyperplane.

Proof. Since $0 \leq H \leq H_f$, the following condition holds

$$|A|^2 H(H - H_f) \leq 0 \leq \frac{H^2}{2}.$$

Moreover, from the proof of Theorem 5 and hypothesis (i), we have

$$0 = |A|^2 H(H - H_f) - \frac{H^2}{2} \leq -\frac{H^2}{2},$$

which implies that $H = 0$ and M^n must be a hyperplane. □

Remark. *The condition $0 \leq H \leq H_f$ implies that M^n is non-compact. Otherwise if M^n was compact, using the same argument as before we would conclude that $H = 0$ which is a contradiction, since there are no compact minimal hypersurfaces in $\mathbb{R}^{n+1}$.*

Using the corollary above, we prove Theorem 6.

Theorem 14. *(Theorem 6) Let M^n be a complete embedded CWMC hypersurface in the Gaussian shrinking Ricci soliton $\mathbb{R}^{n+1}$. Suppose that M^n satisfies the following properties:*
(i) $H \geq 0$;
(ii) $H_f \left(trA^3 (H - H_f) + \frac{|A|^2}{2} \right) \leq 0$;
(iii) $|A| \in L^2_f(M^n)$
Then M^n is either a hyperplane or $S^k_r(0) \times \mathbb{R}^{n-k}$, $1 \leq k \leq n$.

Proof. In the case $H_f \leq 0$, we have $H - H_f \geq 0$ and by Theorem 3 the result follows. Therefore, we only need to study the case $H_f > 0$. If there exists $p \in M^n$ such that $H(p) = H_f$, then by hypothesis (ii) we have

$$\frac{H_f|A|^2(p)}{2} = H_f\left(\text{tr}A^3(H - H_f) + \frac{|A|^2}{2}\right) \leq 0$$

which implies that $|A|^2(p) = 0$. Since $H^2 \leq n|A|^2$, we conclude that $H(p) = 0$ and $H_f = 0$, which is a contradiction. Therefore, either $H - H_f > 0$ or $H - H_f < 0$. The case $H - H_f > 0$ was already proved in Theorem 3. In the other case we have $0 \leq H < H_f$. Since $H^2 \leq n|A|^2$, by hypothesis (iii) we have $H \in L^2_f(M^n)$ and the result follows from Corollary 4.

□

A consequence of the theorem above is as follows.

Corollary 5. *Let M^n be a complete embedded CWMC hypersurface with polynomial volume growth in the Gaussian shrinking Ricci soliton $\mathbb{R}^{n+1}$. If M^n satisfies the following properties:*
(i) $H \geq 0$,
(ii) $H_f\left(trA^3(H - H_f) + \frac{|A|^2}{2}\right) \leq 0$,
then M^n is either a hyperplane or a generalized cylinder $S^k_r(0) \times \mathbb{R}^{n-k}$, $1 \leq k \leq n$.

Proof. The proof is similar to the one proved before. In the case $H_f \leq 0$, we have $H - H_f \geq 0$ and by the theorem proved by Q.Cheng-Wei in [CW18] the result follows. Therefore, we only need to study the case $H_f > 0$. If there exists a point $p \in M^n$ such that $H(p) = H_f$, then by hypothesis (ii) we have

$$\frac{H_f|A|^2(p)}{2} = H_f\left(\text{tr}A^3(H - H_f) + \frac{|A|^2}{2}\right) \leq 0$$

which implies that $|A|^2(p) = 0$. Since $H^2 \leq n|A|^2$, we conclude that $H(p) = 0$ and $H_f = 0$, which is a contradiction. Therefore, either $H - H_f > 0$ or $H - H_f < 0$. The case $H - H_f > 0$ was already proved by Q.Cheng-Wei [CW18]. In the other case we have $0 \leq H < H_f$ and since polynomial volume growth implies that $H \in L^2_f(M^n)$, the result follows by applying Lemma 4. □

Combining Theorem 3 and Theorem 6, we obtain the following result.

Corollary 6. *Let M^n be a complete embedded CWMC hypersurface in the Gaussian shrinking Ricci soliton $\mathbb{R}^{n+1}$ and $\delta \in \{0, 1\}$. If M^n satisfies the following properties:*
(i) $H - \delta H_f \geq 0$,
(ii) $H_f\left(trA^3(H - H_f) + \frac{|A|^2}{2}\right) \leq 0$,
(iii) $|A| \in L^2_f(M^n)$,
then M^n is either a hyperplane or a generalized cylinder $S^k_r(0) \times \mathbb{R}^{n-k}$, $1 \leq k \leq n$.

Assuming a similar condition to the inverse inequality considered in Theorem 5 it is possible to obtain another classification result. In the following, we prove Theorem 7.

Theorem 15. *(Theorem 7) Let M^n be a complete embedded CWMC hypersurface with polynomial volume growth in the Gaussian shrinking Ricci soliton $\mathbb{R}^{n+1}$. If*

$$|A|^2(H - H_f) \geq \frac{H}{2}$$

then M^n is either a hyperplane or a generalized cylinder $S_r^k(0) \times \mathbb{R}^{n-k}$, $1 \leq k \leq n$.

Proof. By the hypothesis we have

$$\Delta_f H = \frac{H}{2} + (H_f - H)|A|^2 \leq 0.$$

In the case M^n is compact, using the maximum principle and Corollary 3, we conclude that $M^n = S_r^n(0)$. In the non-compact case, we shall prove that $\inf_{x \in M^n} H > -\infty$. In fact, if $\inf_{x \in M^n} H = -\infty$, then there exists a sequence $\{p_j\}_{j \in \mathbb{N}}$ in M^n such that $H(p_j) \to -\infty$ when $j \to +\infty$. Therefore, for j large enough $H(p_j) < 0$ and $\frac{H_f}{H(p_j)} < 1$. From hypothesis (i), we have

$$|A|^2(p_j) \leq \frac{1}{2}\left(1 - \frac{H_f}{H(p_j)}\right)^{-1}.$$

Since $\left(1 - \frac{H_f}{H(p_j)}\right)^{-1} \to 1$ when $j \to +\infty$, then the right side of the inequality above is bounded which implies that $|A|^2(p_j)$ is bounded. Therefore, $H^2(p_j)$ is also bounded, but this contradicts the assumption that $H(p_j) \to -\infty$.

Since $\inf_{x \in M^n} H > -\infty$, let us fix $C = \inf_{x \in M^n} H$ and from hypothesis (i), we have

$$\Delta_f (H - C) = \frac{H}{2} + (H_f - H)|A|^2 \leq 0.$$

By the maximum principle either $H - C > 0$ or $H - C = 0$. If $H = C$, by Corollary 3 M^n is either a hyperplane or a generalized cylinder.

If $H - C > 0$, computing $\Delta \log(H - C)$ we get

$$\begin{aligned}
\Delta \log(H - C) &= \operatorname{div}(\nabla \log(H - C)) \\
&= \operatorname{div}\left(\frac{\nabla(H - C)}{H - C}\right) \\
&= \frac{1}{H - C}\Delta(H - C) - |\nabla \log(H - C)|^2.
\end{aligned}$$

Therefore by hypothesis (ii), we obtain

$$\begin{aligned}
\Delta_f \log(H - C) &= \frac{1}{H - C}\Delta_f(H - C) - |\nabla \log(H - C)|^2 \\
&= \frac{H_f - H}{H - C}|A|^2 + \frac{H}{2(H - C)} - |\nabla \log(H - C)|^2 \\
&\leq -|\nabla \log(H - C)|^2.
\end{aligned}$$

Using integration by parts, for $\varphi \in C_0^\infty(M^n)$ we have

$$\begin{aligned}
\int_{M^n} \varphi^2 |\nabla \log(H - C)|^2 e^{-f} &\leq -\int_{M^n} \varphi^2 \Delta_f \log(H - C) e^{-f} \\
&= \int_{M^n} \langle \nabla \log(H - C), \nabla \varphi^2 \rangle e^{-f} \\
&= 2\int_{M^n} \langle \varphi \nabla \log(H - C), \nabla \varphi \rangle e^{-f} \\
&\leq \int_{M^n} \left(\frac{1}{2}\varphi^2 |\nabla \log(H - C)|^2 + 2|\nabla \varphi|^2\right) e^{-f}.
\end{aligned}$$

Therefore

$$\frac{1}{2}\int_{M^n}\varphi^2|\nabla\log(H-C)|^2e^{-f}\leq 2\int_{M^n}|\nabla\varphi|^2e^{-f}.$$

By hypothesis (iii), choosing a sequence as before and using the monotone convergence theorem, we conclude that

$$|\nabla\log(H-C)|^2=0$$

which implies that H is constant, but this contradicts the assumption that $H>\inf_{x\in M^n}H$.

□

A consequence of the Theorem 7 is the following result.

Corollary 7. *Let M^n be a complete embedded CWMC hypersurface with polynomial volume growth in the Gaussian shrinking Ricci soliton $\mathbb{R}^{n+1}$. If the following conditions are satisfied:*
(i) $H\geq 0$,
(ii) $\frac{H}{2}+H_f|A|^2\leq 0$,
then M^n is a hyperplane.

For self-shrinkers, we obtain the following result.

Corollary 8. *Let M^n be a complete embedded CWMC hypersurface with polynomial volume growth in the Gaussian shrinking Ricci soliton $\mathbb{R}^{n+1}$. If*

$$H\left(|A|^2-\frac{1}{2}\right)\geq 0,$$

then M^n is either a hyperplane or a generalized cylinder $S^k_{\sqrt{2k}}(0)\times\mathbb{R}^{n-k}$, $1\leq k\leq n$.

Assuming a condition that only depends on the norm of the second fundamental form, it is possible to obtain another classification theorem. Theorem 8 shows that we can characterize CWMC hypersurfaces by assuming certain bound condition and certain integrability on the norm of the second fundamental form. The next results can be found in [MV21], a joint work in preparation with Matheus Vieira.

Theorem 16. *(Theorem 8) Let M^n be a complete embedded CWMC hypersurface in the Gaussian shrinking Ricci soliton $\mathbb{R}^{n+1}$. If $|A|\in L^q_f(M)$ for some $q\geq 2$ and*

$$|A|\leq\frac{\sqrt{H_f^2+2}-|H_f|}{2},$$

then either M^n is a hyperplane or it is a generalized cylinder $S^k_r(0)\times\mathbb{R}^{n-k}$, where $1\leq k\leq n$.

Proof. By equation (3.2) we have

$$\Delta_f|A|^2=2\left(\frac{1}{2}-|A|^2\right)|A|^2-2H_f\mathrm{tr}A^3+2|\nabla A|^2.$$

Considering $q\geq 2$ as in the hypothesis, we have

$$\begin{aligned}|A|^q\Delta_f|A|^2 &= 2\left(\frac{1}{2}-|A|^2\right)|A|^{q+2}-2|A|^qH_f\mathrm{tr}A^3+2|A|^q|\nabla A|^2\\ &\geq 2\left(\frac{1}{2}-|A|^2\right)|A|^{q+2}-2|H_f||A|^{q+3}+2|A|^q|\nabla A|^2\\ &= |A|^{q+2}(1-2|A|^2-2|H_f||A|)+2|A|^q|\nabla A|^2.\end{aligned}$$

Note that since

$$|A| \le \frac{\sqrt{H_f^2+2}-|H_f|}{2},$$

we have that $1-2|A|^2-2|H_f||A| \ge 0$. Therefore

$$|A|^q \Delta_f |A|^2 \ge 2|A|^q|\nabla A|^2.$$

Let $\varphi \in C_0^\infty(M)$. Integrating by parts we have

$$\begin{aligned}
2\int_M \varphi^2|A|^q|\nabla A|^2 e^{-f} &\le \int_M \varphi^2|A|^q\Delta_f|A|^2 e^{-f}\\
&= -\int_M \langle \nabla\varphi^2|A|^q, \nabla|A|^2\rangle e^{-f}\\
&= -2\int_M q|A|^q\varphi^2|\nabla|A||^2 e^{-f} - 4\int_M \langle |A|^{\frac{q}{2}+1}\nabla\varphi, \varphi|A|^{\frac{q}{2}}\nabla|A|\rangle e^{-f}\\
&\le -2\int_M q|A|^q\varphi^2|\nabla|A||^2 e^{-f} + 4\int_M |A|^{\frac{q}{2}+1}|\nabla\varphi|\varphi|A|^{\frac{q}{2}}|\nabla|A||e^{-f}.
\end{aligned}$$

Here in the last line we use the Cauchy-Schwarz inequality. From the identity $2ab \le \varepsilon a^2 + \frac{1}{\varepsilon}b^2$, choosing $a = |A|^{\frac{q}{2}+1}|\nabla\varphi|$ and $b = \varphi|A|^{\frac{q}{2}}|\nabla|A||$ we have

$$2\int_M \varphi^2|A|^q|\nabla A|^2 e^{-f} \le 2\varepsilon\int_M |A|^{q+2}|\nabla\varphi|^2 e^{-f} + \left(\frac{2}{\varepsilon}-2q\right)\int_M \varphi^2|A|^q|\nabla|A||^2 e^{-f}.$$

Choosing $\varepsilon = \frac{1}{q}$, we get

$$\int_M \varphi^2|A|^q|\nabla A|^2 e^{-f} \le \frac{1}{q}\int_M |A|^{q+2}|\nabla\varphi|^2 e^{-f}.$$

Since $|A|^2 \le C$, we have

$$\int_M \varphi^2|A|^q|\nabla A|^2 e^{-f} \le \frac{C}{q}\int_M |A|^q|\nabla\varphi|^2 e^{-f}.$$

Let us fix $p_0 \in M$ and consider the sequence φ_j such that $\varphi_j = 1$ on $B_j^M(p_0)$, $\varphi_j = 0$ on $M \setminus B_{2j}^M(p_0)$ and $|\nabla\varphi_j| \le \frac{1}{j}$. Using the dominated convergence theorem and hypothesis (ii), we conclude that

$$|A||\nabla A| = 0.$$

Let $\mathcal{A} = \{x \in M; |A|(x) = 0\}$. The set $M \setminus \mathcal{A}$ is open and since

$$|A||\nabla|A|| \le |A||\nabla A| = 0,$$

we see that $|A|$ is constant on $M \setminus \mathcal{A}$. If $\mathcal{A} \neq \emptyset$, then using continuity we conclude that $|A| = 0$, which implies that M^n is a hyperplane. If $\mathcal{A} = \emptyset$, then $|\nabla A| = 0$ and in this case Lawson [Law69] proved that M^n must be a generalized cylinder.

□

A consequence of Theorem 8 is a new result for self-shrinker .

Corollary 9. *Let M^n be a complete embedded self-shrinker in the Gaussian shrinking Ricci soliton $\mathbb{R}^{n+1}$. If $|A| \in L_f^q(M)$ for some $q \ge 2$ and $|A|^2 \le \frac{1}{2}$, then M^n is either a hyperplane passing through the origin or a generalized cylinder $S^k_{\sqrt{2k}}(0) \times \mathbb{R}^{n-k}$, $1 \le k \le n$.*

Some authors have tried to obtain results without the assumption of polynomial volume growth. For self-shrinker submanifolds M^n in $\mathbb{R}^{n+p}$, Q.Cheng-Peng [CP15] used a Omori-Yau principle to prove that if $\sup|A|^2 < 1/2$, then M^n is a hyperplane in $\mathbb{R}^{n+1}$. For codimension 1, by using a result proved by Wei-Wylie [WW09], we generalize Q.Cheng-Peng's result to CWMC hypersurfaces.

Theorem 17. *Let M^n be a complete embedded CWMC hypersurface in $\mathbb{R}^{n+1}$. If*

$$\sup|A| < \frac{\sqrt{H_f^2+2}-|H_f|}{2},$$

then M^n is a hyperplane.

Proof. We estimate the Bakry-Emery-Ricci tensor Ric_f of a CWMC hypersurface. Let $p \in M$ and choose a orthonormal basis of T_pM such that $\nabla_{e_j}e_i(p) = 0$. Let $h_{ij} = A(e_i, e_j)$. We have

$$\begin{aligned}
(Ric_f)_{ij} &= R_{ij} + \nabla_{e_i}\nabla_{e_j} f \\
&= R_{ij} + \bar{\nabla}_{e_i}\bar{\nabla}_{e_j} f + \langle \bar{\nabla} f, A(e_i,e_j)N\rangle \\
&= R_{ij} + \frac{1}{2}\delta_{ij} + \langle \bar{\nabla} f, N\rangle h_{ij} \\
&= -Hh_{ij} - \sum_{l=1}^{n} h_{il}h_{lj} + \langle \bar{\nabla} f, N\rangle h_{ij} + \frac{1}{2}\delta_{ij} \\
&= -Hh_{ij} - \sum_{l=1}^{n} h_{il}h_{lj} + (H-H_f)h_{ij} + \frac{1}{2}\delta_{ij} \\
&= -H_f h_{ij} - \sum_{l=1}^{n} h_{il}h_{lj} + \frac{1}{2}\delta_{ij} \\
&\geq -|H_f||h_{ij}| - \sum_{l=1}^{n} h_{il}h_{lj} + \frac{1}{2}\delta_{ij} \\
&\geq -|H_f||A| - \sum_{l=1}^{n} h_{il}h_{lj} + \frac{1}{2}\delta_{ij}.
\end{aligned}$$

In the fourth line we used the Gauss equation. Since $\sup|A| < \frac{\sqrt{H_f^2+2}-|H_f|}{2}$ we have

$$Ric_f \geq -|A|^2 - |H_f||A| + \frac{1}{2} \geq -(\sup|A|)^2 - |H_f|\sup|A| + \frac{1}{2} > 0.$$

By Theorem 4.1 in [WW09], the condition above implies that M^n has finite weighted volume and applying Theorem 8, we conclude that M^n is a hyperplane. □

A brief comment about the result above is that if the scalar curvature of M^n is non-negative, since

$$R = H^2 - |A|^2$$

an immediate consequence is that if

$$\sup|H| < \frac{\sqrt{H_f^2+2}-|H_f|}{2},$$

then M^n is a hyperplane.

Chapter 4

CWMC hypersurfaces in cylinder shrinking Ricci solitons

We divide this chapter into two sections. First, we study level sets in shrinking Ricci solitons. Then, we prove some geometric results for shrinking Ricci solitons and see related results and applications for cylinder shrinking Ricci solitons. The results in this chapter are found in [MV21], a joint work with Matheus Vieira.

4.1 Level sets in shrinking Ricci solitons

In the first result of this section, we prove that in the case the ambient is a shrinking Ricci soliton with constant scalar curvature, level sets are CWMC hypersurfaces. To do so, recall that Hamilton [Ham93] proved that for gradient Ricci solitons the following holds

$$\left|\bar{\nabla} f\right|^2 + \bar{R} - 2\lambda f = C,$$

where $\bar{R}$ is the scalar curvature of $\bar{M}^{n+1}$ and C is a constant. During this chapter we will always use this notation.

Theorem 18. *Let $\bar{M}^{n+1}$ be a complete gradient shrinking Ricci soliton with constant scalar curvature. Suppose*

$$M^n = \left\{x \in \bar{M}^{n+1} : f(x) = \gamma\right\},$$

where γ is a regular value of f. Then

$$H_f = \frac{n\lambda - 2\lambda\gamma - C}{\sqrt{2\lambda\gamma - \bar{R} + C}},$$

and

$$\gamma = \frac{1}{2\lambda}\left\{\frac{1}{4}\left(-H_f + \sqrt{H_f^2 + 4\left(n\lambda - \bar{R}\right)}\right)^2 + \bar{R} - C\right\}.$$

Moreover $\bar{R} \in \{0, 2\lambda, ..., n\lambda\}$.

Proof. Since $N=\frac{\bar{\nabla} f}{|\bar{\nabla} f|}$ and $\bar{\nabla}\bar{\nabla} f=-\bar{R}ic+\lambda\bar{g}$ we have

$$
\begin{aligned}
A(v,w) &= -\left\langle \bar{\nabla}_v N, w\right\rangle \\
&= -\left\langle \bar{\nabla}_v \frac{\bar{\nabla} f}{|\bar{\nabla} f|}, w\right\rangle \\
&= -\frac{1}{|\bar{\nabla} f|}\bar{\nabla}\bar{\nabla} f(v,w) \\
&= \frac{1}{|\bar{\nabla} f|}\left(\bar{R}ic(v,w)-\lambda\langle v,w\rangle\right).
\end{aligned}
$$

Then

$$
\begin{aligned}
H_f &= H-\left\langle \bar{\nabla} f, N\right\rangle \\
&= \frac{1}{|\bar{\nabla} f|}\left(n\lambda - tr_{M^n}\bar{R}ic\right)-\left\langle \bar{\nabla} f, \frac{\bar{\nabla} f}{|\bar{\nabla} f|}\right\rangle \\
&= \frac{1}{|\bar{\nabla} f|}\left(n\lambda-\bar{R}+\bar{R}ic(N,N)-|\bar{\nabla} f|^2\right).
\end{aligned}
$$

It is well known that if $\bar{M}^{n+1}$ is a gradient Ricci soliton then $\bar{R}ic\left(\bar{\nabla} f\right)=\frac{1}{2}\bar{\nabla}\bar{R}$. Since $\bar{R}$ is constant and $N=\frac{\bar{\nabla} f}{|\bar{\nabla} f|}$ we have $\bar{R}ic(N,N)=0$. We find that

$$
H_f=\frac{1}{|\bar{\nabla} f|}\left(n\lambda-\bar{R}-|\bar{\nabla} f|^2\right).
$$

This proves the first part of the result since $\bar{R}+|\bar{\nabla} f|^2=2\lambda f+C$.

Now we prove the second part of the result. Multiplying by $|\bar{\nabla} f|$ we obtain

$$
|\bar{\nabla} f|^2+H_f|\bar{\nabla} f|-\left(n\lambda-\bar{R}\right)=0.
$$

Solving the quadratic equation we get

$$
|\bar{\nabla} f|=\frac{1}{2}\left(-H_f+\sqrt{H_f^2+4\left(n\lambda-\bar{R}\right)}\right),
$$

or

$$
|\bar{\nabla} f|=\frac{1}{2}\left(-H_f-\sqrt{H_f^2+4\left(n\lambda-\bar{R}\right)}\right).
$$

Claim: The second case does not happen. We will prove the claim later. We conclude that

$$
|\bar{\nabla} f|^2=\frac{1}{4}\left(-H_f+\sqrt{H_f^2+4\left(n\lambda-\bar{R}\right)}\right)^2.
$$

Since $|\bar{\nabla} f|^2=2\lambda f-\bar{R}+C$ we have

$$
f=\frac{1}{2\lambda}\left\{\frac{1}{4}\left(-H_f+\sqrt{H_f^2+4\left(n\lambda-\bar{R}\right)}\right)^2+\bar{R}-C\right\}.
$$

This proves the second part of the result.

Now we prove the claim. We only need to show that

$$
|\bar{\nabla} f|=\frac{1}{2}\left(-H_f-\sqrt{H_f^2+4\left(n\lambda-\bar{R}\right)}\right)\leq 0.
$$

Since $\bar{R}$ is constant on $\bar{M}^{n+1}$ we know that $\bar{R} \in \{0, 2\lambda, ..., n\lambda, (n+1)\lambda\}$ (see [FG16]). We will show that $\bar{R} \in \{0, 2\lambda, ..., n\lambda\}$. Suppose $\bar{R} = (n+1)\lambda$. In this case by [PW09] we see that $\bar{M}^{n+1}$ is Einstein, which implies that

$$\bar{R}ic = \frac{\bar{R}}{n+1}\bar{g} = \lambda\bar{g}.$$

Since $\bar{\nabla}\bar{\nabla}f + \bar{R}ic = \lambda\bar{g}$ we have $\bar{\nabla}\bar{\nabla}f = 0$. This implies that $\left|\bar{\nabla}f\right|$ is constant on $\bar{M}^{n+1}$. Since $2\lambda f = -\left|\bar{\nabla}f\right|^2 - \bar{R} + C$ we see that f is constant on $\bar{M}^{n+1}$ and $\bar{\nabla}f = 0$. This contradicts the fact that M^n is a level set of the potential function. Therefore $\bar{R} \in \{0, 2\lambda, ..., n\lambda\}$. Since $n\lambda - \bar{R} \geq 0$ we have

$$-H_f - \sqrt{H_f^2 + 4\left(n\lambda - \bar{R}\right)} \leq 0.$$

This proves the claim. □

Remark. *Note that when the ambient is a shrinking Ricci soliton with scalar curvature* $\bar{R} = n\lambda$*, there is no level set which is* f*-minimal. Indeed,*

$$H_f = \frac{1}{\left|\bar{\nabla}f\right|}\left(n\lambda - \bar{R} - \left|\bar{\nabla}f\right|^2\right)$$

implies $\left|\bar{\nabla}f\right| = 0$*, a contradiction.*

Using the normalization $\lambda = 1/2$ and $C = \bar{R}$ we have:

Corollary 10. *Let* $\bar{M}^{n+1}$ *be a complete gradient shrinking Ricci soliton with constant scalar curvature. Suppose*

$$M^n = \left\{x \in \bar{M}^{n+1} : f(x) = \gamma\right\},$$

where γ *is a regular value of* f*. Then*

$$H_f = \frac{n - 2\gamma - 2\bar{R}}{2\sqrt{\gamma}},$$

and

$$\gamma = \frac{1}{4}\left(-H_f + \sqrt{H_f^2 + 4\left(\frac{n}{2} - \bar{R}\right)}\right)^2.$$

Moreover $\bar{R} \in \{0, 1, ..., \frac{n}{2}\}$*.*

For 3-dimensional shrinking Ricci solitons we have:

Corollary 11. *Let* $\bar{M}^3$ *be a complete gradient shrinking Ricci soliton with constant scalar curvature. Suppose*

$$M^2 = \left\{x \in \bar{M}^3 : f(x) = \gamma\right\},$$

where γ *is a regular value of* f*. Then*

$$H_f = \frac{1-\gamma}{\sqrt{\gamma}} \text{ in the case } \bar{R} = 0,$$

and

$$H_f = -\sqrt{\gamma} \text{ in the case } \bar{R} = 1.$$

For 4-dimensional shrinking Ricci solitons we have:

Corollary 12. *Let $\bar{M}^4$ be a complete gradient shrinking Ricci soliton with constant scalar curvature. Suppose*

$$M^3 = \left\{x \in \bar{M}^4 : f(x) = \gamma\right\},$$

where γ is a regular value of f. Then

$$H_f = \frac{3-2\gamma}{2\sqrt{\gamma}}, \quad \textit{in the case} \;\; \bar{R} = 0,$$

$$H_f = \frac{1-2\gamma}{2\sqrt{\gamma}}, \quad \textit{in the case} \;\; \bar{R} = 1,$$

and

$$H_f = -\sqrt{\gamma}, \quad \textit{in the case} \;\; \bar{R} = \frac{3}{2}.$$

Another consequence of Theorem 18 is the following.

Corollary 13. *Let $r > 0$. Then $S_r^{n-k}(0) \times S^k_{\sqrt{2(k-1)}}(0)$ is a CWMC hypersurface in the cylinder shrinking Ricci soliton $\mathbb{R}^{n+1-k} \times S^k_{\sqrt{2(k-1)}}(0)$ and $r = -H_f + \sqrt{H_f^2 + 2(n-k)}$.*

Proof. By Example 2.4 we have $\lambda = \frac{1}{2}$, $\bar{R} = \frac{k}{2}$ and $C = \bar{R}$ (since $C - \bar{R} = \left|\bar{\nabla} f\right|^2 - 2\lambda f$). We have

$$S_r^{n-k}(0) \times S^k_{\sqrt{2(k-1)}}(0) = \left\{(x,y) \in \mathbb{R}^{n+1-k} \times S^k_{\sqrt{2(k-1)}}(0); f(x,y) = \gamma\right\},$$

where $\gamma = \frac{r^2}{4}$. By Theorem 18 we have

$$\gamma = \frac{1}{4}\left(-H_f + \sqrt{H_f^2 + 2(n-k)}\right)^2.$$

This proves the result. □

4.2 Geometric properties for CWMC in shrinking Ricci solitons

In this section we prove Theorem 19, Theorem 21, Theorem 23 and related results.

Let us define

$$D^{\pm}(c_1, c_2) = \left\{x \in \bar{M}^{n+1}; f(x) < \Gamma^{\pm}(x)\right\},$$

where

$$\Gamma^{\pm}(x) = \frac{1}{2\lambda}\left\{\frac{1}{4}\left(\pm c_1 + \sqrt{c_1^2 + 4c_2}\right)^2 + \bar{R}(x) - C\right\}$$

and c_1 and c_2 are constants such that $c_1^2 + 4c_2 \geq 0$.

Note that $D^+(c_1, c_2)$ and $D^-(c_1, c_2)$ are open subsets of $\bar{M}^{n+1}$. In this chapter we will always use the above notations.

Theorem 19. *Let $\bar{M}^{n+1}$ be a shrinking Ricci soliton and let M^n be a CWMC hypersurface immersed in $\bar{M}^{n+1}$ with finite weighted volume.*

(a) Suppose $tr_{M^n}\bar{\nabla}\bar{\nabla} f \geq a$ for some $a > 0$. If M^n lies in $\overline{D^-(|H_f|, a)}$ then $M^n \subseteq \partial D^-(|H_f|, a)$.

(b) Suppose $tr_{M^n}\bar{\nabla}\bar{\nabla} f \leq b$ for some $b > 0$. If M^n lies outside $D^+(|H_f|, b)$ then $M^n \subseteq \partial D^+(|H_f|, b)$.

Proof. Fact (a). Let u be a function on M^n. If u is bounded from above and $\Delta_f u \geq 0$ then u is constant.

Fact (b). Let u be a function on M^n. If u is bounded from below and $\Delta_f u \leq 0$ then u is constant.

Both facts follow from Corollary 1 in [CY75] since M^n has finite weighted volume. We remark that the extension of these results to smooth metric measure spaces is straightforward.

We have

$$\nabla\nabla f\,(u,v) = \bar{\nabla}\bar{\nabla} f\,(u,v) + \left\langle \bar{\nabla} f, (\bar{\nabla}_u v)^{\perp} \right\rangle.$$

We see that

$$\begin{aligned}\Delta_f f &= tr_{M^n}\bar{\nabla}\bar{\nabla} f + \left\langle \bar{\nabla} f, \vec{H} \right\rangle - \langle \nabla f, \nabla f\rangle \\ &= tr_{M^n}\bar{\nabla}\bar{\nabla} f + \left\langle \bar{\nabla} f, \vec{H} + (\bar{\nabla} f)^{\perp} \right\rangle - \langle \bar{\nabla} f, \bar{\nabla} f\rangle \\ &= tr_{M^n}\bar{\nabla}\bar{\nabla} f + \left\langle \bar{\nabla} f, \vec{H}_f \right\rangle - \left|\bar{\nabla} f\right|^2.\end{aligned}$$

Proof of Item (a). Since $tr_{M^n}\bar{\nabla}\bar{\nabla} f \geq a$ we have

$$\Delta_f f \geq a - |H_f|\left|\bar{\nabla} f\right| - \left|\bar{\nabla} f\right|^2.$$

Claim: We have

$$a - |H_f|\left|\bar{\nabla} f\right| - \left|\bar{\nabla} f\right|^2 \geq 0 \;\; on \;\; M^n,$$

and equality holds if and only if $M^n \subseteq \partial D^-(|H_f|, a)$. We will prove this claim later. By the claim we have

$$\Delta_f f \geq a - |H_f|\left|\bar{\nabla} f\right| - \left|\bar{\nabla} f\right|^2 \geq 0.$$

Since $f \leq \Gamma^-$ on M^n, by Fact (a) we see that f is constant on M^n. We conclude that

$$a - |H_f|\left|\bar{\nabla} f\right| - \left|\bar{\nabla} f\right|^2 = 0 \;\; on \;\; M^n.$$

By the claim we have $M^n \subseteq \partial D^-(|H_f|, a)$. Now we prove the claim. We have

$$\begin{aligned} & a - |H_f|\left|\bar{\nabla} f\right| - \left|\bar{\nabla} f\right|^2 \geq 0 \;\; on \;\; M^n \\ \iff & \left(\left|\bar{\nabla} f\right| + \frac{|H_f|}{2}\right)^2 \leq \frac{H_f^2}{4} + a \;\; on \;\; M^n \\ \iff & \left|\bar{\nabla} f\right| \leq \frac{1}{2}\left(-|H_f| + \sqrt{H_f^2 + 4a}\right) \;\; on \;\; M^n.\end{aligned}$$

Since $\left|\bar{\nabla} f\right|^2 = 2\lambda f - \bar{R} + C$ and

$$2\lambda\Gamma^- - \bar{R} + C = \frac{1}{4}\left(-|H_f| + \sqrt{H_f^2 + 4a}\right)^2,$$

we have

$$\begin{aligned} M^n \subset \overline{D^-(|H_f|, a)} &\iff f \leq \Gamma^- \;\; on \;\; M^n \\ &\iff 2\lambda f - \bar{R} + C \leq 2\lambda\Gamma^- - \bar{R} + C \;\; on \;\; M^n \\ &\iff \left|\bar{\nabla} f\right|^2 \leq \frac{1}{4}\left(-|H_f| + \sqrt{H_f^2 + 4a}\right)^2 \;\; on \;\; M^n \\ &\iff \left|\bar{\nabla} f\right| \leq \frac{1}{2}\left(-|H_f| + \sqrt{H_f^2 + 4a}\right) \;\; on \;\; M^n.\end{aligned}$$

In the last line we used the fact that $a > 0$. We conclude that

$$a - |H_f| \left|\bar{\nabla} f\right| - \left|\bar{\nabla} f\right|^2 \geq 0 \ \ on \ \ M^n \iff M^n \subset \overline{D^-(|H_f|, a)}.$$

Moreover equality holds in the left hand side if and only if $M^n \subseteq \partial D^-(|H_f|, a)$. This proves the claim.

Proof of Item (b). Since $tr_{M^n}\bar{\nabla}\bar{\nabla} f \leq b$ we have

$$\Delta_f f \leq b + |H_f| \left|\bar{\nabla} f\right| - \left|\bar{\nabla} f\right|^2.$$

Claim. We have

$$b + |H_f| \left|\bar{\nabla} f\right| - \left|\bar{\nabla} f\right|^2 \leq 0 \ \ on \ \ M^n,$$

and equality holds if and only if $M^n \subseteq \partial D^+(|H_f|, b)$. We will prove this claim later. By the claim we have

$$\Delta_f f \leq b + |H_f| \left|\bar{\nabla} f\right| - \left|\bar{\nabla} f\right|^2 \leq 0.$$

Since $f \geq \Gamma^+$ on M^n, by Fact (b) we see that f is constant on M^n. We conclude that

$$b + |H_f| \left|\bar{\nabla} f\right| - \left|\bar{\nabla} f\right|^2 = 0 \ \ on \ \ M^n.$$

By the claim we have $M^n \subseteq \partial D^+(|H_f|, b)$. Now we prove the claim. We have

$$\begin{aligned} & b + |H_f| \left|\bar{\nabla} f\right| - \left|\bar{\nabla} f\right|^2 \leq 0 \ \ on \ \ M^n \\ \iff & \left(\left|\bar{\nabla} f\right| - \frac{|H_f|}{2}\right)^2 \geq \frac{H_f^2}{4} + b \ \ on \ \ M^n \\ \iff & \left|\bar{\nabla} f\right| \geq \frac{1}{2}\left(|H_f| + \sqrt{H_f^2 + 4b}\right) \ \ on \ \ M^n. \end{aligned}$$

Since $\left|\bar{\nabla} f\right|^2 = 2\lambda f - \bar{R} + C$ and

$$2\lambda\Gamma^+ - \bar{R} + C = \frac{1}{4}\left(|H_f| + \sqrt{H_f^2 + 4b}\right)^2,$$

we have

$$\begin{aligned} M^n \subset \bar{M}^{n+1} \setminus D^+(|H_f|, b) & \iff f \geq \Gamma^+ \ \ on \ \ M^n \\ & \iff 2\lambda f - \bar{R} + C \geq 2\lambda\Gamma^+ - \bar{R} + C \ \ on \ \ M^n \\ & \iff \left|\bar{\nabla} f\right|^2 \geq \frac{1}{4}\left(|H_f| + \sqrt{H_f^2 + 4b}\right)^2 \ \ on \ \ M^n \\ & \iff \left|\bar{\nabla} f\right| \geq \frac{1}{2}\left(|H_f| + \sqrt{H_f^2 + 4b}\right) \ \ on \ \ M^n. \end{aligned}$$

In the last line we used the fact that $b \geq 0$. We conclude that

$$b + |H_f| \left|\bar{\nabla} f\right| - \left|\bar{\nabla} f\right|^2 \leq 0 \ \ on \ \ M^n \iff M^n \subset \bar{M}^{n+1} \setminus D^+(|H_f|, b).$$

Moreover equality holds in the left hand side if and only if $M^n \subseteq \partial D^+(|H_f|, b)$. □

Note that we can state Theorem 19 in a different way using the fact that

$$tr_{M^n}\bar{\nabla}\bar{\nabla} f = n\lambda - \bar{R} + \bar{R}ic\,(N, N)\,,$$

where N is the unit normal vector of M^n.

As an aplication of Theorem 9 we have following result for CWMC hypersurfaces in a cylinder shrinking Ricci soliton.

Theorem 20. *(Theorem 9) Let M^n be a complete CWMC hypersurface properly immersed in the cylinder shrinking Ricci soliton $\mathbb{R}^{n+1-k} \times S^k_{\sqrt{2(k-1)}}(0)$, where $n \geq 3$ and $2 \leq k \leq n-1$.*

(a) If M^n lies inside the closed product $\bar{B}^{n+1-k}_r(0) \times S^k_{\sqrt{2(k-1)}}(0)$ with $r = -|H_f| + \sqrt{H_f^2 + 2(n-k)}$, then $H_f \geq 0$ and $M^n = S^{n-k}_r(0) \times S^k_{\sqrt{2(k-1)}}(0)$.

(b) M^n cannot lie outside the product $B^{n+1-k}_r(0) \times S^k_{\sqrt{2(k-1)}}(0)$ with $r = |H_f| + \sqrt{H_f^2 + 2(n+1-k)}$.

Proof. By Example 2.4 we have $\lambda = \frac{1}{2}$ and $C = \bar{R}$ (since $C - \bar{R} = |\bar{\nabla} f|^2 - 2\lambda f$). We have $\bar{\nabla}\bar{\nabla} f = \frac{1}{2} g_{R^{n+1-k}}$ and

$$tr_{M^n} \bar{\nabla}\bar{\nabla} f = \bar{\Delta} f - \bar{\nabla}\bar{\nabla} f(N,N) = \frac{n+1-k}{2} - \bar{\nabla}\bar{\nabla} f(N,N).$$

In particular

$$\frac{n-k}{2} \leq tr_{M^n} \bar{\nabla}\bar{\nabla} f \leq \frac{n+1-k}{2}.$$

For $D^-(|H_f|, a)$ with $a = \frac{n-k}{2}$ we have

$$\Gamma^- = \frac{1}{4}\left(-|H_f| + \sqrt{H_f^2 + 2(n-k)}\right)^2,$$

and for $D^+(|H_f|, b)$ with $b = \frac{n-k+1}{2}$ we have

$$\Gamma^+ = \frac{1}{4}\left(|H_f| + \sqrt{H_f^2 + 2(n+1-k)}\right)^2.$$

Since M^n is properly immersed and $\bar{\nabla}\bar{\nabla} f(N,N) = \frac{1}{2} g_{\mathbb{R}^{n-k+1}}(N,N) \geq 0$ we know that M^n has finite weighted volume (see Remark 4.2).

Proof of Item (a). We have $D^-(|H_f|, a) = B^{n+1-k}_r(0) \times S^k_{\sqrt{2(k-1)}}(0)$ and $\partial D^-(|H_f|, a) = S^{n-k}_r(0) \times S^k_{\sqrt{2(k-1)}}(0)$ where $r = -|H_f| + \sqrt{H_f^2 + 2(n-k)}$. By Item (a) of Theorem 19 we have $M^n \subseteq S^{n-k}_r(0) \times S^k_{\sqrt{2(k-1)}}(0)$. Since M^n is complete we have $M^n = S^{n-k}_r(0) \times S^k_{\sqrt{2(k-1)}}(0)$. On the other hand, by Corollary 13 we have $r = -H_f + \sqrt{H_f^2 + 2(n-k)}$, so $H_f \geq 0$.

Proof of Item (b). We have $D^+(|H_f|, b) = B^{n-k+1}_r(0) \times S^k_{\sqrt{2(k-1)}}(0)$ and $\partial D^+(|H_f|, b) = S^{n-k}_r(0) \times S^k_{\sqrt{2(k-1)}}(0)$ where $r = |H_f| + \sqrt{H_f^2 + 2(n+1-k)}$. Suppose M^n lies outside $B^{n+1-k}_r(0) \times S^k_{\sqrt{2(k-1)}}(0)$. By Item (b) of Theorem 19 we have $M^n \subseteq S^{n-k}_r(0) \times S^k_{\sqrt{2(k-1)}}(0)$. Since M^n is complete we have $M^n = S^{n-k}_r(0) \times S^k_{\sqrt{2(k-1)}}(0)$. On the other hand, by Corollary 13 we have $r = -H_f + \sqrt{H_f^2 + 2(n-k)}$. Therefore

$$\begin{aligned} -H_f + \sqrt{H_f^2 + 2(n-k)} &= |H_f| + \sqrt{H_f^2 + 2(n+1-k)} \\ &> |H_f| + \sqrt{H_f^2 + 2(n-k)}. \end{aligned}$$

Hence

$$-H_f > |H_f|,$$

which is a contradiction.

$\square$

An application of Theorem 19 to the Gaussian shrinking soliton ambient space is the following.

Corollary 14. *Let M^n be a complete CWMC hypersurface properly immersed in the Gaussian shrinking Ricci soliton $\mathbb{R}^{n+1}$.*

(a) If M^n lies inside $\bar{B}_r^{n+1}(0)$ with $r = -|H_f| + \sqrt{H_f^2 + 2n}$, then $H_f \geq 0$ and $M^n = S_r^n(0)$.

(b) If M^n lies outside $B_r^{n+1}(0)$ with $r = |H_f| + \sqrt{H_f^2 + 2n}$, then $H_f \leq 0$ and $M^n = S_r^n(0)$.

In the next results, we will see that by assuming an upper on the mean curvature, one can conclude that M^n lies in certain level set. We prove the following result in the case the ambient is a shrinking Ricci soliton.

Theorem 21. *Let $\bar{M}^{n+1}$ be a shrinking Ricci soliton and let M^n be a compact CWMC hypersurface immersed in $\bar{M}^{n+1}$. Suppose $tr_{M^n}\bar{\nabla}\bar{\nabla} f \geq a$ for some $a > 0$. Assume that $f(p) = \sup_{M^n} f$ is a regular value of f. If*

$$H \leq \frac{2H_f - |H_f| + \sqrt{H_f^2 + 4a}}{2},$$

then $M^n \subseteq \partial D^-(|H_f|, a)$.

Proof. Since M^n is a compact hypersurface, the potential function achieves its maximum in some $p \in M$. From the fact that the gradient of f points in the direction of greatest increase and $\nabla f(p) = 0$, we see that $\bar{\nabla} f(p)$ and $N(p)$ have the same direction. Therefore

$$\begin{aligned} H_f + |\bar{\nabla} f|(p) &= H_f + \langle \bar{\nabla} f(p), N(p) \rangle \\ &= H(p). \end{aligned}$$

Using the normalization equation and the hypothesis on the mean curvature, we get

$$\begin{aligned} \sqrt{2\lambda f(p) + C - \bar{R}} = |\bar{\nabla} f|(p) &\leq \frac{2H_f - |H_f| + \sqrt{H_f^2 + 4a}}{2} - H_f \\ &= \frac{-|H_f| + \sqrt{H_f^2 + 4a}}{2}. \end{aligned}$$

Therefore

$$f \leq f(p) \leq \frac{1}{2\lambda}\left\{ \left(\frac{-|H_f| + \sqrt{H_f^2 + 4a}}{2} \right)^2 + \bar{R} - C \right\},$$

which implies that M^n lies in $\overline{D^-(|H_f|, a)}$. By Theorem 19, we conclude that $M^n \subseteq \partial D^-(|H_f|, a)$. $\square$

This is a new result even for f-minimal hypersurfaces in cylinder shrinking Ricci solitons.

Corollary 15. *Let M^n be a compact f-minimal hypersurface immersed in the cylinder shrinking Ricci soliton $\mathbb{R}^{n-k+1} \times S^k_{\sqrt{2(k-1)}}(0)$, where $n \geq 3$ and $2 \leq k \leq n-1$. If*

$$H \leq \frac{\sqrt{2(n-k)}}{2},$$

then $M^n = S_r^{n-k}(0) \times S^k_{\sqrt{2(k-1)}}(0)$, where $r = \sqrt{2(n-k)}$.

Theorem 22. *(Theorem 10) Let M^n be a compact CWMC hypersurface immersed in the cylinder shrinking Ricci soliton $\mathbb{R}^{n-k+1} \times S^k_{\sqrt{2(k-1)}}(0)$, where $n \geq 3$ and $2 \leq k \leq n-1$. If $H_f \geq 0$ and*

$$H \leq \frac{H_f + \sqrt{H_f^2 + 2(n-k)}}{2},$$

then $M^n = S^{n-k}_r(0) \times S^k_{\sqrt{2(k-1)}}(0)$, where $r = -H_f + \sqrt{H_f^2 + 2(n-k)}$.

Proof. Take $a = \frac{n-k}{2}$ as in the proof of Corollary 19 Item (a). The result follows from Theorem 21.

□

By assuming a lower bound on the mean curvature, it is possible to obtain a result for the case the ambient is a smooth measure metric space.

Theorem 23. *Let $(\bar{M}^{n+1}, \bar{g}, f)$ be a a smooth measure metric space and let M^n be a hypersurface immersed in $\bar{M}^n$ with finite weighted volume. Suppose $tr_M \bar{\nabla}\bar{\nabla} f \leq b$ for some $b > 0$. If*

$$H \geq \frac{H_f + \sqrt{H_f^2 + 4b}}{2},$$

then $M^n \subseteq f^{-1}(\gamma)$ for some $\gamma \in \mathbb{R}$.

Proof. We compute

$$\begin{aligned}
\Delta f &= \mathrm{tr}_M \bar{\nabla}\bar{\nabla} f - \langle \bar{\nabla} f, N\rangle H \\
&= \mathrm{tr}_M \bar{\nabla}\bar{\nabla} f + (H_f - H)H \\
&\leq b + (H_f - H)H \\
&= \frac{H_f^2 + 4b}{4} - \left(H - \frac{H_f}{2}\right)^2 \leq 0.
\end{aligned}$$

Let $\varphi \in C_0^\infty(M)$. Integrating by parts, we have

$$\begin{aligned}
\int_M \varphi^2 |\nabla f|^2 e^{-f} &\leq \int_M \varphi^2 (|\nabla f|^2 - \Delta f) e^{-f} \\
&= -\int_M \varphi^2 \Delta_f f e^{-f} \\
&= \int_M \langle \nabla \varphi^2, \nabla f\rangle e^{-f} \\
&= 2\int_M \langle \nabla \varphi, \varphi \nabla f\rangle e^{-f} \\
&\leq \int_M \left[2|\nabla \varphi|^2 + \frac{1}{2}\varphi^2 |\nabla f|^2\right] e^{-f}.
\end{aligned}$$

Therefore

$$\frac{1}{2}\int_M \varphi^2 |\nabla f|^2 e^{-f} \leq 2\int_M |\nabla \varphi|^2 e^{-f}.$$

Fix $p_0 \in M^n$ and consider a sequence $\varphi_j \in C_0^\infty(M^n)$ such that $\varphi_j = 1$ on $B_j^M(p_0)$, $\varphi_j = 0$ on $M^n \setminus B_{2j}^M(p_0)$ and $|\nabla \varphi_j| \leq \frac{1}{j}$. By the dominated convergence theorem we have

$$\int_M |\nabla f|^2 e^{-f} = 0,$$

which implies that f is constant on M^n. □

Using Theorem 23, Remark 4.2 and Theorem 18 we obtain a new result for ambient spaces which are shrinking Ricci solitons with constant scalar curvature. Note that taking $b = \frac{n}{2}$, $\lambda = \frac{1}{2}$, $\bar{R} = 0$ and $C = 0$ in the next result we recover Theorem 1.2 in [COW16].

Corollary 16. *Let $\bar{M}^{n+1}$ be a shrinking Ricci soliton with constant scalar curvature and let M^n be a complete CWMC hypersurface properly immersed in $\bar{M}^{n+1}$. Suppose $tr_M \bar{\nabla}\bar{\nabla} f \leq b$ for some $b > 0$. If*

$$H \geq \frac{H_f + \sqrt{H_f^2 + 4b}}{2},$$

then $M^n \subseteq f^{-1}(\gamma)$ for some $\gamma > 0$. Moreover, if $f^{-1}(\gamma)$ is connected and complete then $M^n = f^{-1}(\gamma)$ and

$$\gamma = \frac{1}{2\lambda}\left\{\frac{1}{4}\left(-H_f + \sqrt{H_f^2 + 4(n\lambda - \bar{R})}\right)^2 + \bar{R} - C\right\}.$$

Proof. The first part is an immediate consequence of Theorem 23 and Remark 4.2. Suppose $f^{-1}(\gamma)$ is connected and complete. Then $M^n = f^{-1}(\gamma)$. Since $\bar{M}^{n+1}$ has constant scalar curvature we have

$$\bar{R}ic(N, N) = 0,$$

which implies that $\mathrm{tr}_M \bar{\nabla}\bar{\nabla} f = n\lambda - \bar{R}$. Therefore $b = n\lambda - \bar{R}$ and by Theorem 18 we have

$$\gamma = \frac{1}{2\lambda}\left\{\frac{1}{4}\left(-H_f + \sqrt{H_f^2 + 4(n\lambda - \bar{R})}\right)^2 + \bar{R} - C\right\}.$$

□

Theorem 24. *(Theorem 11) Let M^n be a complete CWMC hypersurface properly immersed in the cylinder shrinking Ricci soliton $\mathbb{R}^{n+1-k} \times S^k_{\sqrt{2(k-1)}}(0)$, where $n \geq 3$ and $2 \leq k \leq n-1$. Then*

$$\inf H < \frac{H_f + \sqrt{H_f^2 + 2(n+1-k)}}{2}.$$

Proof. Take $b = \frac{n-k+1}{2}$ as in the proof of Corollary 9 Item (b). The result follows from Corollary 16.

□

This is a new result even for f-minimal hypersurfaces.

Corollary 17. *Let M^n be a complete f-minimal hypersurface properly immersed in the cylinder shrinking Ricci soliton $\mathbb{R}^{n+1-k} \times S^k_{\sqrt{2(k-1)}}(0)$, where $n \geq 3$ and $2 \leq k \leq n-1$. Then*

$$\inf H < \frac{\sqrt{2(n+1-k)}}{2}.$$

Remark. *When $\bar{\nabla}\bar{\nabla} f\,(N, N) \geq k$ for some $k \in \mathbb{R}$ we know that the assumptions of finite weighted volume, polynomial volume growth and properly immersed are equivalent to each other (see Theorem 1.3 in [CVZ19]). In particular, if $\bar{M}^{n+1}$ has constant scalar curvature and $tr_{M^n} \bar{\nabla}\bar{\nabla} f \leq b$ for some $b > 0$, then M^n has finite weighted volume if and only if M^n is properly immersed. This follows easily from the equation*

$$\begin{aligned}(n+1)\lambda &= \bar{R} + \bar{\Delta} f \\ &= \bar{R} + tr_{M^n}\bar{\nabla}\bar{\nabla} f + \bar{\nabla}\bar{\nabla} f(N, N).\end{aligned}$$

CHAPTER **5**

Appendix

In the first section we prove a relation between the exponential volume growth condition and properness for submanifolds in the Gaussian shrinking Ricci soliton $\mathbb{R}^{n+1}$ with weighted mean curvature of at most linear growth. Then, in the second section we prove some non-existence type theorems for the case where the ambient is a gradient Ricci soliton.

5.1 Volume estimates

For self-shrinkers, Ding-Xin [DX+13] proved that the properness condition implies Euclidean volume growth. X.Cheng-Zhou [CZ13] proved that properness, Euclidean volume growth, polynomial volume growth and finite weighted volume are all equivalent for self-shrinkers. Alencar and Rocha [AR18] showed that $|\vec{H}_f| < \infty$ and finite weighted volume imply properness. Also, a consequence from one of their results is that for $f = |x|^2/4$, $\sup\langle \vec{H}_f, \overline{\nabla} f\rangle < \infty$ and properness imply finite weighted volume and polynomial volume growth. Recently, X.Cheng, Vieira and Zhou [CVZ19] proved that for hypersurfaces in $\mathbb{R}^{n+1}$ with the norm of the weighted mean curvature bounded, properness is equivalent to polynomial volume growth. In [AM20], we prove the following:

Theorem 25. *For any complete n-dimensional immersed submanifold M^n in the Gaussian shrinking Ricci soliton $\mathbb{R}^{n+p}$, $p \geq 1$, satisfying*

$$|\vec{H}_f| \leq a_1 r + a_0 \text{ on } M^n \cap B_r(0),\ \forall r > 0$$

for some $0 \leq a_0$ and $0 \leq a_1 < \frac{1}{2}$, the following statements are equivalent:
(i) M^n properly immersed on $\mathbb{R}^{n+p}$;
(ii) There exist constants $C, \overline{a}_0, \overline{a}_1, \overline{a}_2$, with $\overline{a}_2 < \frac{1}{4}$, such that

$$V(B_r(0) \cap M^n) \leq C e^{\overline{a}_2 r^2 + \overline{a}_1 r + \overline{a}_0};$$

(iii)$\int_{M^n} e^{-f} < \infty$.

Let us consider $D_r \subset M$ as the level set

$$D_r = \{x \in M; 2\sqrt{f} < r\},$$

and $V(r)$ the volume of $\overline{D}_r$.

Theorem 26. *Let $(M^n, g, e^{-f}dv)$ be a complete non-compact smooth measure metric space, with $f : M \to \mathbb{R}$ a proper function on M. If $|\nabla f|^2 \leq f$ and $\Delta_f f + f \leq a_2 r^2 + a_1 r + a_0$ on D_r for all $r > 0$, where a_0, a_1, are constants and $0 \leq a_2 < \frac{1}{4}$, then $Vol_f(M) < \infty$ and for $\varepsilon > 0$ arbitrary*

$$V(r) \leq Ce^{\varepsilon(a_2 r^2 + a_1 r + a_0) + \frac{r^2}{4e^{\varepsilon}}}. \tag{5.1}$$

Proof. Let us define

$$I(t) = \frac{1}{t^{\kappa(r)}} \int_{\bar{D}_r} e^{-\frac{f}{t}} dv$$

for all $t > 0$, where $\kappa(r) = a_2 r^2 + a_1 r + a_0$. Since f is proper, I is well defined.
Computing the derivative of I, we get

$$I'(t) = t^{-\kappa(r)-1} \int_{\bar{D}_r} e^{-\frac{f}{t}} \left(\frac{f}{t} - \kappa(r) \right) dv. \tag{5.2}$$

On the other hand

$$\begin{aligned}
\int_{\bar{D}_r} \operatorname{div}\left(e^{-\frac{f}{t}} \nabla f\right) dv &= \int_{\bar{D}_r} e^{-\frac{f}{t}} \left(\Delta f - \frac{|\nabla f|^2}{t} \right) dv \\
&\leq \int_{\bar{D}_r} e^{-\frac{f}{t}} \left(|\nabla f|^2 - f + \kappa(r) - \frac{|\nabla f|^2}{t} \right) dv \\
&\leq \int_{\bar{D}_r} e^{-\frac{f}{t}} \left(\frac{(t-1)}{t} f - f + \kappa(r) \right) dv, \quad t \geq 1 \\
&= \int_{\bar{D}_r} e^{-\frac{f}{t}} \left(\kappa(r) - \frac{f}{t} \right) dv \\
&= -I'(t) t^{\kappa(r)+1}.
\end{aligned}$$

Therefore, we have

$$I'(t) \leq -t^{-\kappa(r)-1} \int_{\bar{D}_r} \operatorname{div}\left(e^{-\frac{f}{t}} \nabla f\right) dv.$$

For every r such that $\frac{r^2}{4}$ is a regular value of f, D_r has smooth boundary. By the Stokes theorem, we get

$$\begin{aligned}
I'(t) &\leq -t^{-\kappa(r)-1} \int_{\partial D_r} \left\langle e^{-\frac{f}{t}} \nabla f, \frac{\nabla f}{|\nabla f|} \right\rangle dv \\
&\leq -t^{-\kappa(r)-1} \int_{\partial D_r} e^{-\frac{f}{t}} |\nabla f| dv \leq 0.
\end{aligned}$$

Integrating $I'(t)$ over t, from 1 to e^{ε}, where $\varepsilon > 0$ is arbitrary, we obtain $I(e^{\varepsilon}) \leq I(1)$, that is

$$e^{-\varepsilon\kappa(r)} \int_{\bar{D}_r} e^{-\frac{f}{e^{\varepsilon}}} dv \leq \int_{\bar{D}_r} e^{-f} dv. \tag{5.3}$$

By the monotone convergence theorem, the inequality above holds for any $r > 0$. Since $2\sqrt{f} \leq r$ on $\bar{D}_r$, we have

$$e^{-\varepsilon\kappa(r)} e^{-\frac{r^2}{4e^{\varepsilon}}} \int_{\bar{D}_r} dv \leq \int_{\bar{D}_r} e^{-f} dv. \tag{5.4}$$

Moreover,

$$\int_{\bar{D}_r} e^{-f} dv - \int_{\bar{D}_{r-1}} e^{-f} dv = \int_{\bar{D}_r \setminus \bar{D}_{r-1}} e^{-f} dv \leq e^{-\frac{(r-1)^2}{4}} \int_{\bar{D}_r} dv. \tag{5.5}$$

Combining (5.4) and (5.5), we get

$$\int_{\bar{D}_r} e^{-f}dv - \int_{\bar{D}_{r-1}} e^{-f}dv \leq e^{\varepsilon\kappa(r)+\frac{r^2}{4e^{\varepsilon}}-\frac{(r-1)^2}{4}} \int_{\bar{D}_r} e^{-f}dv. \tag{5.6}$$

Since $\kappa(r) = a_2r^2 + a_1r + a_0$ and $a_2 < \frac{1}{4}$ there exist $r_0 \in \mathbb{R}$ such that for $r \geq r_0$ and ε_0 sufficiently small

$$e^{\varepsilon_0(a_2r^2+a_1r+a_0)+\frac{r^2}{4e^{\varepsilon_0}}-\frac{(r-1)^2}{4}} < e^{-r}.$$

From (5.6), we have

$$\int_{\bar{D}_r} e^{-f}dv \leq \frac{1}{1-e^{-r}} \int_{\bar{D}_{r-1}} e^{-f}dv.$$

Then for any integer N

$$\int_{\bar{D}_{r_0+N}} e^{-f}dv \leq \left(\prod_{i=0}^{N} \frac{1}{1-e^{-r_0-i}}\right) \int_{\bar{D}_{r_0-1}} e^{-f}dv < \infty,$$

which implies that $\int_M e^{-f}dv < +\infty$. Moreover, from (5.4) we obtain

$$e^{-\varepsilon\kappa(r)}e^{-\frac{r^2}{4e^{\varepsilon}}} \int_{\bar{D}_r} dv \leq \int_{\bar{D}_r} e^{-f}dv \leq \int_M e^{-f}dv < \infty.$$

Therefore

$$V(r) \leq Ce^{\varepsilon\kappa(r)+\frac{r^2}{4e^{\varepsilon}}}.$$

□

□

Remark. *An immediate consequence is that under the same hypothesis of Theorem 26, for $\varepsilon = 0$ we have the following volume estimate*

$$V(r) \leq Ce^{\frac{r^2}{4}}.$$

In the following result, we obtain a volume estimate for submanifolds with weighted mean curvature of at most linear growth. In particular, we obtain an explicit estimate for the volume of self-expanders and translating solitons.

Corollary 18. *Let M^n be a complete submanifold immersed in the Gaussian shrinking Ricci soliton $\mathbb{R}^{n+p}$ and such that*

$$|\vec{H}_f| \leq a_1r + a_0 \text{ on } M^n \cap B_r(0), \ \forall r > 0$$

for some $0 \leq a_0$ and $0 \leq a_1 < \frac{1}{2}$. If M^n is properly immersed on $\mathbb{R}^{n+p}$, then

$$V(B_r(0) \cap M^n) \leq Ce^{\varepsilon\left(\frac{a_1r^2+a_0r+n}{2}\right)+\frac{r^2}{4e^{\varepsilon}}}$$

for $\varepsilon > 0$ arbitrary.

Proof. Let us verify that f satisfies the conditions of Theorem 26. Indeed

$$f - |\nabla f|^2 = \frac{|x^\perp|^2}{4} \geq 0$$

and

$$\begin{aligned} \Delta_f f + f &= \frac{n}{2} + \langle \vec{H}, \overline{\nabla} f \rangle - |\nabla f|^2 + f \\ &= \frac{n}{2} + \langle \vec{H}_f - (\overline{\nabla} f)^\perp, (\overline{\nabla} f)^\perp \rangle - |\nabla f|^2 + f \\ &= \frac{n}{2} + \langle \vec{H}_f, (\overline{\nabla} f)^\perp \rangle. \end{aligned}$$

Therefore

$$\Delta_f f + f \leq \frac{a_1 r^2 + a_0 r + n}{2}$$

on $M^n \cap B_r(0)$. Since M^n is properly immersed, it follows that f is proper on M^n. Applying Theorem 26, we obtain $\int_{M^n} e^{-f} < \infty$ and

$$V(B_r(0) \cap M^n) \leq C e^{\varepsilon \left(\frac{a_1 r^2 + a_0 r + n}{2} \right) + \frac{r^2}{4e^\varepsilon}}$$

for $\varepsilon > 0$ arbitrary. □

□

Using the corollary above, we prove Theorem 25.

Proof of Theorem 25 From Corollary 18, we get that (i) implies (ii). To prove that (ii) implies (iii), we provide the following estimate

$$\begin{aligned} \int_{M^n} e^{\frac{-|x|^2}{4}} dv &\leq \sum_{j=1}^{\infty} \int_{M^n \cap B_j(0) \setminus B_{j-1}(0)} e^{\frac{-|x|^2}{4}} dv \\ &\leq \sum_{j=1}^{\infty} e^{\frac{-(j-1)^2}{4}} V(M^n \cap B_j(0)) \\ &\leq C \sum_{j=1}^{\infty} e^{\frac{-(j-1)^2}{4}} e^{\bar{a}_2 r^2 + \bar{a}_1 r + \bar{a}_0} \\ &= C \sum_{j=1}^{\infty} e^{\frac{(4\bar{a}_2 - 1) j^2 + (2 + 4\bar{a}_1) j + (4\bar{a}_0 - 1)}{4}}. \end{aligned}$$

Since $\bar{a}_2 < \frac{1}{4}$, the right side of the inequality converges. Therefore

$$\int_{M^n} e^{\frac{-|x|^2}{4}} dv < \infty.$$

Finally, we prove that (iii) implies (i). Indeed, suppose M^n is not proper. Then there exists r_0 such that $\bar{B}_{r_0}(0) \cap M^n$ is not compact on M^n. Thus, for a positive constant a, there exists a sequence $\{p_k\}$ on $\bar{B}_{r_0}(0) \cap M^n$ such that $d_M(p_k, p_j) \geq a$. Therefore, $B^M_{\frac{a}{2}}(p_k) \cap B^M_{\frac{a}{2}}(p_j) = \emptyset$, where B^M is the geodesic ball on M^n. Choosing $0 < a < \min\{2r_0, \frac{n}{(2a_1+1)r_0 + a_0}\}$, for all $p \in B^M_{\frac{a}{2}}(p_k)$

$$|p| \leq |p - p_k| + |p_k| \leq d_M(p, p_k) + |p_k| \leq 2r_0$$

which implies that $B^M_{\frac{a}{2}}(p_k) \subset B_{2r_0}(0)$ for all k.
For all $p \in M^n \cap B_{2r_0}(0)$

$$\begin{aligned} |\vec{H}|(p) &\leq |\vec{H}_f(p)| + \frac{|p^\perp|}{2} \\ &\leq (2a_1+1)r_0 + a_0. \end{aligned}$$

Considering $r_k : B^M_{\frac{a}{2}}(p_k) \to \mathbb{R}$, where $r_k(x) = |x - p_k|$, we have

$$\begin{aligned} \Delta r_k^2 &= 2n + \langle \vec{H}, \nabla r_k^2 \rangle \\ &\geq 2n - 2|\vec{H}|r_k \\ &\geq 2n - 2(a_0 + (2a_1+1)r_0)r_k. \end{aligned}$$

By the divergence theorem, since $a \leq \frac{n}{(2a_1+1)r_0+a_0}$, for all $0 < r < \frac{a}{2}$

$$\begin{aligned} \int_{B^M_r(p_k)} [2n - 2a_0 r_k - 2(2a_1+1)r_0 r_k] dv &\leq \int_{B^M_r(p_k)} \Delta r_k^2 dv \\ &= \int_{\partial B^M_r(p_k)} \langle \nabla r_k^2, N \rangle dv \\ &\leq 2rA(r) \end{aligned}$$

where N is the outward normal vector of $\partial B^M_r(p_k)$ and $A(r)$ the area of $\partial B^M_r(p_k)$. Using the co-area formula, we obtain

$$\begin{aligned} \int_0^r [n - a_0 s - (2a_1+1)r_0 s] A(s) ds &= \int_0^r \int_{d_M(x,p_k)=s} [n - a_0 r_k - (2a_1+1)r_0 r_k] dv \\ &= \int_{B^M_r(p_k)} [n - a_0 r_k - (2a_1+1)r_0 r_k] dv \\ &\leq rA(r). \end{aligned}$$

Therefore,

$$\left(\frac{n}{r} - a_0 - (2a_1+1)r_0\right) \leq \frac{V'(r)}{V(r)}.$$

Integrating from $\varepsilon > 0$ to r, we obtain

$$\log\left(\frac{r}{\varepsilon}\right)^n - (a_0 + (2a_1+1)r_0)(r - \varepsilon) \leq \log\frac{V(r)}{V(\varepsilon)}.$$

Thus

$$r^n e^{-(a_0+2(a_1+1)r_0)(r-\varepsilon)} \frac{V(\varepsilon)}{\varepsilon^n} \leq V(r).$$

Since

$$\lim_{\varepsilon \to 0} \frac{V(\varepsilon)}{\varepsilon^n} = \omega_n,$$

for any $0 < r \leq \frac{a}{2}$, we have

$$V(r) \geq r^n \omega_n e^{-(a_0+2(a_1+1)r_0)r}.$$

Finally, considering that $B^M_{\frac{a}{2}}(p_k) \cap B^M_{\frac{a}{2}}(p_j) = \emptyset$ for $k \neq j$, $B^M_{\frac{a}{2}}(p_k) \subset B_{2r_0}(0)$ for all k, and the inequality obtained above, we have

$$\begin{aligned}\int_{M^n} e^{-\frac{|x|^2}{4}} dv &\geq \sum_{k=1}^{\infty} \int_{B^M_{\frac{a}{2}}(p_k)} e^{-\frac{|x|^2}{4}} dv \\ &\geq e^{-r_0^2} \sum_{k=1}^{\infty} V\left(\frac{a}{2}\right) = +\infty,\end{aligned}$$

which is a contradiction.

□

An immediate consequence of the theorem above is as follows.

Corollary 19. *For any complete n-dimensional CWMC hypersurface M^n in the Gaussian shrinking Ricci soliton $\mathbb{R}^{n+1}$, the following statements are equivalent:*
(i) M^n properly immersed on $\mathbb{R}^{n+1}$;
(ii) There exist constants a_2, a_1, a_0, with $a_2 < \frac{1}{4}$, and $C > 0$ such that

$$V(B_r(0) \cap M^n) \leq C e^{a_2 r^2 + a_1 r + a_0};$$

(iii) $\int_{M^n} e^{-f} < \infty$.

5.2 Further results in Gradient Ricci solitons

Here we prove some non-existence type theorems for gradient Ricci solitons.

Using similar computations as in the proof of Theorem 18 we show that the level sets cannot be f-minimal when the ambient space is a steady or expanding Ricci soliton with constant scalar curvature.

Proposition 1. *Let $\bar{M}^{n+1}$ be a complete gradient Ricci soliton with constant scalar curvature. If there exists a level set of the potential function f which is f-minimal, then $\bar{M}^{n+1}$ is a gradient shrinking Ricci soliton.*

Proof. Since $\bar{M}^{n+1}$ is a gradient Ricci soliton we have

$$\bar{R}ic(\bar{\nabla} f) = \frac{1}{2}\bar{\nabla}\bar{R}.$$

Since $N = \frac{\bar{\nabla} f}{|\bar{\nabla} f|}$ and the assumption that the scalar curvature $\bar{R}$ is constant we find that

$$\bar{R}ic(N,N) = \frac{1}{2}\frac{\langle \bar{\nabla}\bar{R}, \bar{\nabla} f\rangle}{|\bar{\nabla} f|} = 0.$$

From Theorem 18 we have

$$H_f = \frac{1}{|\bar{\nabla} f|}\left(n\lambda - 2\lambda f - C\right).$$

From the assumption that M^n is f-minimal and from the normalization we have

$$\begin{aligned} 0 &= n\lambda - 2\lambda f - C \\ &= n\lambda - 2\lambda f - C + \bar{R} - \bar{R} \\ &= n\lambda - \bar{R} - |\bar{\nabla} f|^2. \end{aligned}$$

First suppose $\bar{M}^{n+1}$ is a steady soliton. Zhang proved in [Zha09] that the scalar curvature $\bar{R}$ is always non-negative and from the equation above we conclude that $|\bar{\nabla} f| = 0$, which is a contradiction with the fact that M^n is a level of the potential function.
Now suppose $\bar{M}^{n+1}$ is an expanding Ricci soliton. By [FG16], the scalar curtature satisfies $\bar{R} \in \{(n+1)\lambda, n\lambda, ..., 2\lambda, 0\}$. Using the same argument as in the proof of Lemma 18, $\bar{R} \geq n\lambda$. But from the equation above this implies that $|\bar{\nabla} f| = 0$, which is a contradiction with the fact that M^n is a level of the potential function. □

Note that the same result is true if we replace the assumption of constant scalar curvature by the assumptions that $\bar{R} \geq n\lambda$ and the Ricci curvature of $\bar{M}^{n+1}$ vanishes on the normal bundle of M^n.

In [CL13], Cao-Li showed that there are no compact self-expanders in the expanding Ricci soliton $(\mathbb{R}^{n+1}, \bar{g}_{can}, f = -|x|^2/4)$. Thus, the above result was expected for expanding Ricci solitons with a proper potential function.

Considering a hypersurface in general, the following result holds.

Proposition 2. *Let M^n be a hypersurface in a gradient Ricci soliton $\bar{M}^{n+1}$. If the following condition holds,*

$$n\lambda - \bar{R} + \bar{R}ic(N,N) + (H_f - H)H \leq 0$$

with the inequality strict at some point of M^n, then M^n cannot be compact.

Proof. By computing the Hessian of f, we have

$$\nabla\nabla f = \bar{\nabla}\bar{\nabla} f + \langle \bar{\nabla} f, A(\ ,\ )N\rangle$$

Therefore, the Laplacian

$$\begin{aligned} \Delta f &= \mathrm{tr}\bar{\nabla}\bar{\nabla} f - \langle \bar{\nabla} f, N\rangle H \\ &= n\lambda - \mathrm{tr}_M \bar{R}ic + (H_f - H)H \\ &= n\lambda - \bar{R} + \bar{R}ic(N,N) + (H_f - H)H \leq 0. \end{aligned}$$

If M^n is a compact hypersurface, by the divergence theorem we get that

$$n\lambda - \bar{R} + \bar{R}ic(N,N) + (H_f - H)H = 0$$

which is a contradiction since there exist a point $p \in M$ such that

$$(n\lambda - \bar{R} + \bar{R}ic(N,N) + (H_f - H)H)(p) < 0.$$

□

As immediate consequences we have the following.

Corollary 20. *Let M^n be a hypersurface in the steady Ricci soliton $\mathbb{R}^{n+1}$. If the following condition holds,*

$$(H_f - H)H \leq 0$$

with the inequality strict at some point of M^n, then M^n cannot be compact.

Corollary 21. *Let M^n be a hypersurface in the expanding Ricci soliton $\mathbb{R}^{n+1}$. If the following condition holds,*

$$(H_f - H)H \leq \frac{n}{2}$$

with the inequality strict at some point of M^n, then M^n cannot be compact.

Notice that the last corollary generalizes the result proved by Cao-Li.

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